Answer on Question #39341 – Math – Abstract Algebra

Question. How many onto functions are there from an n-element set to an m-element set?

Solution. Let $N = \{1, \ldots, n\}$, and $M = \{1, \ldots, m\}$ and $Q(n, m)$ be the number of surjective functions $N \to M$. For every finite set X denote by $|X|$ the number of elements in X.

If $n < m$, then there is no such function, that is

$$Q(n, m) = 0, \qquad n < m.$$

Thus assume that $n \geq m$.

Notice that in general, the number of all functions $N \to M$ is m^n. Let $T(n, m, k)$ be the number of functions $f : N \to M$ whose image consists exactly of k elements, so $|f(N)| = k$. Then

$$Q(n, m) = T(n, m, m).$$

Let $X \subset M$ be a subset such that $|X| = k$ and $0 < k < m$. Then the number of function $f : N \to M$ such that $f(N) = X$ is equal to

$$Q(n, k).$$

The number of k-element subsets of M is equal to the binomial coefficient

$$C_k^m = \frac{m!}{k!(m-k)!}.$$

Therefore

$$T(n, m, k) = C_k^m Q(n, k).$$

Hence the number of surjective functions $N \to M$ is equal to

$$\begin{aligned} Q(n, m) &= m^n - T(n, m, m-1) - T(n, m, m-2) - \cdots - T(n, m, 1) \\ &= m^n - C_{m-1}^m Q(n, m-1) - C_{m-2}^m Q(n, m-2) - \cdots - C_1^m Q(n, 1). \end{aligned}$$

Now let us compute $Q(n, m)$ for small values of m to find a general rule for $Q(n, m)$.

If $m = 1$, then there exists only one function $N \to M = \{1\}$, so

$$Q(n, 1) = 1.$$

Suppose $m = 2$, and let $f : N \to M = \{1, 2\}$ be any surjective function. Then we obtain a partition of N into two non-empty sets:

$$f^{-1}(1), \qquad f^{-1}(2).$$

In fact, $f^{-1}(2) = N \setminus f^{-1}(1)$, and so f is uniquely determined by the set $f^{-1}(1)$. Conversely, any subset $X \subset N$ distinct from two sets $\varnothing$ and X determines a unique function:

$$f : N \to M, \qquad f(X) = 1, \qquad , f(N \setminus X) = 2.$$

The number of all subsets in N is 2^n, whence the number all subsets of N distinct from $\varnothing$ and X, that is the number of onto functions $N \to M$ is $2^n - 2$. Thus

$$Q(n, 2) = 2^n - 2.$$

Let $m = 3$. Then

$$\begin{aligned} Q(n, 3) &= 3^n - C_2^3 Q(n, 2) - C_1^3 Q(n, 1) \\ &= 3^n - 3(2^n - 2) - 3 = 3^n - 3 \cdot 2^n + 3. \end{aligned}$$

For $m = 4$ we have

$$\begin{aligned} Q(n,4) &= 4^n - C_3^4 Q(n,3) - C_2^4 Q(n,2) - C_1^4 Q(n,1) \\ &= 4^n - 4(3^n - 3\cdot 2^n + 3) - 6(2^n - 2) - 4 \\ &= 4^n - 4\cdot 3^n + 12\cdot 2^n - 12 - 6\cdot 2^n + 12 - 4 \\ &= 4^n - 4\cdot 3^n + 6\cdot 2^n - 4. \end{aligned}$$

We see that for $m = 1, 2, 3, 4$ the following formula holds true:

$$\begin{aligned} Q(n,1) &= C_1^1 \cdot 1^n, \\ Q(n,2) &= C_2^2 \cdot 2^n - C_1^2 \cdot 1^n, \\ Q(n,3) &= C_3^3 \cdot 3^n - C_2^3 \cdot 2^n + C_1^3 \cdot 1^n, \\ Q(n,4) &= C_4^4 \cdot 4^n - C_3^4 \cdot 3^n + C_2^4 \cdot 2^n - C_1^4 \cdot 1^n. \end{aligned}$$

Let us prove that

$$Q(n,m) = m^n - C_{m-1}^m (m-1)^n + C_{m-2}^m (m-2)^n - C_{m-3}^m (m-3)^n \cdots$$

For $i \in M$ let

$$F_i = \{f : N \to M \mid f(X) \subset M \setminus \{i\}\}$$

be the set of functions $f : N \to M$ whose image does not contain i and F be the set of all functions $f : N \to M$.

Then the number of all surjective functions $f : N \to M$ is equal to

$$\left|\bigcap_{i=1}^{m} (F \setminus F_i)\right| = \left|F \setminus \bigcup_{i=1}^{m} F_i\right| |F| - \left|\bigcup_{i=1}^{m} F_i\right|. \tag{1}$$

Notice that for any family $\{F_i\}$ of subsets of a set F we have that

$$\left|\bigcup_{i=1}^{m} F_i\right| = \sum_i |F_i| - \sum_{i_1<i_2} |F_{i_1} \cap F_{i_2}| + \sum_{i_1<i_2<i_3} |F_{i_1} \cap F_{i_2} \cap F_{i_3}| + \cdots \\ \cdots + (-1)^{m-1} |F_1 \cap F_2 \cap \cdots \cap F_m|. \tag{2}$$

Indeed, suppose $x \in \bigcup_{i=1}^m F_i$ belongs exactly to k subsets, say $F_1, \ldots, F_k$. Then x is counted in

$$\sum_i |F_i|$$

$k = C_1^m$ times, in

$$\sum_{i_1<i_2} |F_{i_1} \cap F_{i_2}|$$

C_2^m times, in

$$\sum_{i_1<i_2<i_3} |F_{i_1} \cap F_{i_2} \cap F_{i_3}|$$

C_3^m times and so on. Therefore in right hand side x will be counted

$$a = C_1^m - C_2^m + C_3^m - \cdots + (-1)^{m-1} C_m^m$$

times. But from the identity

$$\begin{aligned} 0 = (1-1)^m &= C_0^m - C_1^m + C_2^m - C_3^m + \cdots + (-1)^{m-1} C_m^m \\ &= C_0^m - a = 1 - a, \end{aligned}$$

we get that

$$a = C_1^m - C_2^m + C_3^m - \cdots + (-1)^{m-1} C_m^m = 1.$$

This proves (2).

Now turning back to (1) notice that

$$|F_i| = (m-1)^n.$$

Also

$$|F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}|$$

is the set of all functions $N \to M \setminus \{i_1, \ldots, i_k\}$, and so this number is $(m-k)^n$. Hence

$$\sum_{i_1 < i_2 < \cdots < i_k} |F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}| = C_k^m (m-k)^n.$$

Therefore

$$\left|\bigcup_{i=1}^{m} F_i\right| = \sum_i |F_i| - \sum_{i_1<i_2} |F_{i_1} \cap F_{i_2}| + \sum_{i_1<i_2<i_3} |F_{i_1} \cap F_{i_2} \cap F_{i_3}| + \cdots$$

$$\cdots + (-1)^{m-1} |F_1 \cap F_2 \cap \cdots \cap F_m| =$$

$$= C_{m-1}^m (m-1)^n - C_{m-2}^m (m-2)^n + C_{m-3}^m (m-3)^n \cdots$$

Thus

$$Q(n,m) = \left|\bigcap_{i=1}^{m} (F \setminus F_i)\right| = |F| - \left|\bigcup_{i=1}^{m} F_i\right| =$$

$$= m^n - C_{m-1}^m (m-1)^n + C_{m-2}^m (m-2)^n - C_{m-3}^m (m-3)^n \cdots$$

Answer. The number of onto functions from an n-element set to an m-element set is

$$Q(n,m) = 0, \qquad n < m$$

and

$$Q(n,m) = m^n - C_{m-1}^m (m-1)^n + C_{m-2}^m (m-2)^n - C_{m-3}^m (m-3)^n \cdots$$

for $n \geq m$.