Answer on Question #39216 – Math – Trigonometry

Question: If the ratio of three sides of a triangle is $a: b: c = 7: 8: 9$, then show that $\cos A : \cos B : \cos C = 14: 11: 6$.

Solution. Let us apply the law of cosines three times, using each angle once.

Start with $\cos A$ and thus the side a:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$-2bc \cos A = a^2 - b^2 - c^2$$

$$-2 \cos A = \frac{a^2}{bc} - \frac{b^2}{bc} - \frac{c^2}{bc}$$

$$-2 \cos A = \frac{a}{b} \cdot \frac{a}{c} - \frac{b}{c} - \frac{c}{b}$$

Now note that we are given each of the ratios $\frac{a}{b}, \frac{a}{c}, \frac{b}{c}$. Therefore,

$$-2 \cos A = \frac{7}{8} \cdot \frac{7}{9} - \frac{8}{9} - \frac{9}{8}$$

$$-2 \cos A = \frac{7^2 - 8^2 - 9^2}{8 \cdot 9} = \frac{49 - 64 - 81}{72}$$

$$2 \cos A = \frac{96}{72}$$

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$$\cos A = \frac{2}{3}.$$

Similarly, for $\cos B$ we have

$$b^2 = a^2 + c^2 - 2ac \cos B.$$

Making the same transformations as above, we arrive at

$$-2 \cos B = \frac{b}{a} \cdot \frac{b}{c} - \frac{a}{c} - \frac{c}{a}$$

$$-2 \cos B = \frac{8}{7} \cdot \frac{8}{9} - \frac{7}{9} - \frac{9}{7}$$

and find

$$\cos B = \frac{11}{21}.$$

Finally, for $\cos C$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$-2 \cos C = \frac{c}{a} \cdot \frac{c}{b} - \frac{a}{b} - \frac{b}{a}$$

$$-2 \cos C = \frac{9}{7} \cdot \frac{9}{8} - \frac{7}{8} - \frac{8}{7}$$

and

$$\cos C = \frac{2}{7}.$$

Now we only need to find the ratio of obtained cosines:

$$\cos A : \cos B : \cos C = \frac{2}{3} : \frac{11}{21} : \frac{2}{7}$$

Bring all the fractions on the left side of the equality to a common denominator:

$$\cos A : \cos B : \cos C = \frac{14}{21} : \frac{11}{21} : \frac{6}{21},$$

or

$$\cos A : \cos B : \cos C = 14 : 11 : 6.$$