Answer on Question#39113 – Math - Discrete Mathematics

How many solutions are there to the equation
x1+x2+x3+x4+x5=21,
where xi, i=1, 2, 3, 4, 5, is a non negative integer such that
b) xi≥1 for i=1, 2, 3, 4, 5?
c) 0≤x1≤11?

Solution:
b) We make a substitution: $y_i = x_i - 1, i = 1, \dots, 5$ and we obtain the following equation

$$y_1 + y_2 + y_3 + y_4 + y_5 = 21 - 5 = 16.$$

And $y_i \geq 0$ for i=1, 2, 3, 4, 5.
There is a 1-1 correspondence between the solutions and reorderings of 16 ones and 4 zeros (y_1 is the number of ones before the first zero, $y_2$the number of ones between the first and the second zero, and so on).
Hence the answer is

$$C_{16+4}^4 = C_{20}^4 = 4845.$$

c) Similarly to the case b) we make a substitution: $y_i = x_i - 1, i = 2, \dots, 5$ and we obtain the following equation

$$x_1 + y_2 + y_3 + y_4 + y_5 = 21 - 4 = 17.$$

And $y_i \geq 0$ for i= 2, 3, 4, 5.
If x1 is fixed then the number of solutions for the other variables is

$$C_{17-x_1+3}^3 = C_{20-x_1}^3$$

Since as in b), the solutions are in 1-1 correspondence with reorderings of $29 - x_1$ ones and 4 zeros.
The possible values for x1 are 0, 1,…, 11 and hence the answer is

$$C_{20}^3 + C_{19}^3 + \cdots + C_{10}^3 + C_9^3 = \frac{20!}{3!\,17!} + \frac{19!}{3!\,16!} + \cdots + \frac{10!}{3!\,7!} + \frac{9!}{3!\,6!} =$$

$$= \frac{9!}{3!\,6!}\left(\frac{10 * \dots * 20}{7 * \dots * 17} + \frac{10 * \dots * 19}{7 * \dots * 16} + \cdots + \frac{10}{7} + 1\right) =$$

$$= \frac{9!}{3!\,6!}\left(\frac{18 * 19 * 20}{7 * 8 * 9} + \frac{17 * 18 * 19}{7 * 8 * 9} + \frac{16 * 17 * 18}{7 * 8 * 9} + \frac{15 * 16 * 17}{7 * 8 * 9} + \frac{14 * 15 * 16}{7 * 8 * 9} + \frac{13 * 14 * 15}{7 * 8 * 9} + \frac{12 * 13 * 14}{7 * 8 * 9} + \frac{11 * 12 * 13}{7 * 8 * 9} + \frac{10 * 11 * 12}{7 * 8 * 9} + \frac{10 * 11}{7 * 8} + \frac{10}{7} + 1\right)$$

$$= 84 * \left(\frac{6840 + 5814 + 4896 + 4080 + 3360 + 2730 + 2184 + 1716 + 1320 + 990 + 720 + 504}{504}\right)$$

$$= \frac{84 * 35154}{504} = 5859.$$

Answer: b) 4845
c) 5859