Answer on Question #39003 - <Math> - <Algebra>.

Solve the following ordinary differential equations; $xdx + ydy + 4y^3(x^2 + y^2)dy$

Solution:

We can rewrite our equation:

$$\mathrm{x\,dx} + \left(4(\mathrm{x}^2 + \mathrm{y(x)}^2)\mathrm{y(x)}^3 + \mathrm{y(x)}\right)\mathrm{dy(x)} = 0$$

Make the replacement, let $\mathrm{R(x,y) = x}$ and $\mathrm{S(x,y)} = 4\mathrm{y}^3(\mathrm{x}^2+\mathrm{y}^2)+\mathrm{y}$. This is not exact equation, because $\frac{\partial \mathrm{R(x,y)}}{\partial \mathrm{y}} = 0 \neq 8\mathrm{xy}^3 = \frac{\partial \mathrm{S(x,y)}}{\partial \mathrm{x}}$.

Next step we have to find an integrating factor $\mu(y)$ such that

$\mu(\mathrm{y})\mathrm{R(x,y)dx} + \mu(\mathrm{y})\mathrm{S(x,y)dy(x)} = 0$ is exact.

This means that $\frac{\partial}{\partial y}\left(\mu(\mathrm{y})\mathrm{R(x,y)}\right) = \frac{\partial}{\partial x}(\mu(\mathrm{y})\mathrm{S(x,y)})$:

$$\frac{\partial \mu(y)}{\partial y}x = 8y^3\mu(y)x$$

Isolate $\mu(y)$ to the left-hand side:

$$\frac{\frac{\partial \mu(\mathrm{y})}{\partial \mathrm{y}}}{\mu(\mathrm{y})} = 8y^3$$

Integrate both sides with respect to y:

$$\log(\,\mu(y)) = 2y^4$$

Then we take exponentials of the both sides:

$$\mu(y) = e^{2y^4}$$

Go back to our expression and multiply it:

$$\left(4(\mathrm{x}^2 + \mathrm{y(x)}^2)\mathrm{y(x)}^3 + \mathrm{y(x)}\right)\frac{\mathrm{dy(x)}}{\mathrm{dx}} + x = 0 \text{ by } \mu(y(x)):$$

$$e^{2y(x)^4}x + \left(e^{2y(x)^4}\left(4(\mathrm{x}^2 + \mathrm{y(x)}^2)\mathrm{y(x)}^3 + \mathrm{y(x)}\right)\right)\frac{\mathrm{dy(x)}}{\mathrm{dx}} = 0$$

Let $\mathrm{P(x,y) = xe^{2y^4}}$ and $\mathrm{Q(x,y)} = \mathrm{e}^{2\mathrm{y}^4}(4\mathrm{y}^3(\mathrm{x}^2+\mathrm{y}^2)+\mathrm{y})$ this is an exact equation, because $\frac{\partial \mathrm{P(x,y)}}{\partial \mathrm{y}} = 8\mathrm{xe}^{2\mathrm{y}^4}\mathrm{y}^3 = \frac{\partial \mathrm{Q(x,y)}}{\partial \mathrm{x}}$

Define $\mathrm{f(x,y)}$ such that $\frac{\partial \mathrm{f(x,y)}}{\partial \mathrm{x}} = \mathrm{P(x,y)}$ and $\frac{\partial \mathrm{f(x,y)}}{\partial y} = \mathrm{Q(x,y)}$:

Then the solution will be given by $\mathrm{f(x,y)} = C_1$, where C_1 is an arbitrary constant. Integrate $\frac{\partial \mathrm{f(x,y)}}{\partial \mathrm{x}}$ with respect to x in order to find $\mathrm{f(x,y)}$:

$$f(x,y) = \int e^{2y^4} x dx = \frac{1}{2} e^{2y^4} x^2 + g(y)$$

Where g(y) is an arbitrary function of y.

Differentiate f(x, y) with respect to y in order to find the g(y):

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{2} e^{2y^4} x^2 + g(y)\right) = 4e^{2y^4} y^3 x^2 + \frac{\partial g(y)}{\partial y}$$

Substitute into $\frac{\partial f(x,y)}{\partial y} = Q(x,y)$:

$$4e^{2y^4} y^3 x^2 + \frac{\partial g(y)}{\partial y} = e^{2y^4}(y + 4y^3(x^2 + y^2))$$

Then we solve for $\frac{\partial g(y)}{\partial y}$:

$$\frac{\partial g(y)}{\partial y} = e^{2y^4}(4y^5 + y)$$

Integrate $\frac{\partial g(y)}{\partial y}$ with respect to y:

$$g(y) = \int e^{2y^4}(4y^5 + y) dy = \frac{1}{2} e^{2y^4} y^2$$

Substitute $g(y)$ into f(x, y):

$$f(x,y) = \frac{1}{2} e^{2y^4} x^2 + \frac{1}{2} e^{2y^4} y^2$$

Eventually we get the solution $f(x,y) = C_1$ (as we note earlier).

The solution is $C_1 = \frac{1}{2} e^{2y^4} x^2 + \frac{1}{2} e^{2y^4} y^2$

$$C_1 = \frac{1}{2} x^2 e^{2y(x)^4} + \frac{1}{2} e^{2(x)^4} y(x)^2$$

Answer: $C_1 = \frac{1}{2} x^2 e^{2y(x)^4} + \frac{1}{2} e^{2(x)^4} y(x)^2$