Answer on Question#39001 – Math - Other

The Method of Frobenius is a way to find an infinite series solution for a second-order ordinary differential equation of the form

$$z^2u'' + p(z)zu' + q(z)u = 0$$

in the vicinity of the regular singular point $z = 0$.

We can divide by z^2 to obtain a differential equation of the form

$$u'' + \frac{p(z)}{z}u' + \frac{q(z)}{z^2}u = 0$$

We have the differential equation:

$$y'' + \frac{1}{2x}y' + y = 0$$

The Method of Frobenius tells us that we can seek a power series solution of the form:

$$y(x) = \sum_{k=0}^{\infty} A_k x^{k+r}$$

Differentiating:

$$y'(x) = \sum_{k=0}^{\infty} (k+r)A_k x^{k+r-1}$$

$$y''(x) = \sum_{k=0}^{\infty} (k+r-1)A_k x^{k+r-2}$$

Substitute it into equation:

$$\sum_{k=0}^{\infty} (k+r-1)A_k x^{k+r-2} + \frac{1}{2x}\sum_{k=0}^{\infty} (k+r)A_k x^{k+r-1} + \sum_{k=0}^{\infty} A_k x^{k+r-1}$$

$$= \sum_{k=0}^{\infty} (k+r-1)A_k x^{k+r-2} + \sum_{k=0}^{\infty} \frac{k+r}{2} A_k x^{k+r-2} + \sum_{k=0}^{\infty} A_k x^{k+r-1}$$

$$= \sum_{k=0}^{\infty} (k+r-1)A_k x^{k+r-2} + \sum_{k=0}^{\infty} \frac{k+r}{2} A_k x^{k+r-2} + \sum_{k-1=0}^{\infty} A_{k-1} x^{k+r-2}$$

$$= \sum_{k=0}^{\infty} (k+r-1)A_k x^{k+r-2} + \sum_{k=0}^{\infty} \frac{k+r}{2} A_k x^{k+r-2} + \sum_{k=1}^{\infty} A_{k-1} x^{k+r-2}$$

$$= \sum_{k=0}^{\infty} \left(k + r - 1 + \frac{k}{2} + \frac{r}{2}\right) A_k x^{k+r-2} + \sum_{k=1}^{\infty} A_{k-1} x^{k+r-2}$$

$$= \left(r - 1 + \frac{r}{2}\right) A_0 x^{r-2} + \sum_{k=1}^{\infty} \left(k + r - 1 + \frac{k}{2} + \frac{r}{2}\right) A_k x^{k+r-2} + \sum_{k=1}^{\infty} A_{k-1} x^{k+r-2}$$

$$= \left(\frac{3r}{2} - 1\right) A_0 x^{r-2} + \sum_{k=1}^{\infty} \left(\left(\frac{3k}{2} + \frac{3r}{2} - 1\right) A_k + A_{k-1}\right) x^{k+r-2} = 0$$

From $\frac{3r}{2} - 1 = 0$ we get $r = \frac{2}{3}$. Using this root, we set the coefficient of x^{k+r-2} to be zero (for it to be a solution), which gives us:

$$\left(\frac{3k}{2} + 1 - 1\right) A_k + A_{k-1} = \frac{3k}{2} A_k + A_{k-1} = 0$$

Hence we have the recurrence relation:

$$A_k = -\frac{2}{3k} A_{k-1}$$

Then

$$A_0 = A_1 = 1\ (arbitrary), \qquad A_2 = -\frac{2}{3} A_1 = -\frac{2}{3}, \qquad A_3 = -\frac{2}{3 * 2} * \left(-\frac{2}{3}\right) = \frac{2^2}{3^2 2!}, \dots$$

Or

$$A_k = \frac{(-1)^{k-1} 2^{k-1}}{3^{k-1} k!}$$

If we will have some initial conditions, we can either solve the recurrence entirely or obtain a solution in power series form.

Answer:

$$y_r(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k-1} 2^{k-1}}{3^{k-1} k!} x^{k+r}.$$