Answer on Question#38998 - Math - Differential Calculus|Equation

Question: Show that

$$u(x,t) = \sin(\omega x)\, e^{-\omega^2 c^2 t}$$

is a solution of the one-dimensional heat equation.

Solution. Recall the one-dimensional heat equation:

$$u_t = c^2 u_{xx}.$$

We need to find the above derivatives (u_t, u_{xx}) of the given function and substitute them into the equation.

$$u_t(x,t) = \sin(\omega x) \cdot e^{-\omega^2 c^2 t} \cdot (-\omega^2 c^2) = -\omega^2 c^2 \sin(\omega x)\, e^{-\omega^2 c^2 t},$$

$$u_x(x,t) = \cos(\omega x) \cdot \omega \cdot e^{-\omega^2 c^2 t} = \omega \cos(\omega x)\, e^{-\omega^2 c^2 t},$$

$$u_{xx}(x,t) = -\sin(\omega x) \cdot \omega^2 \cdot e^{-\omega^2 c^2 t} = -\omega^2 \sin(\omega x)\, e^{-\omega^2 c^2 t}.$$

Therefore, we see that

$$u_t(x,t) = \sin(\omega x) \cdot e^{-\omega^2 c^2 t} \cdot (-\omega^2 c^2) = -\omega^2 c^2 \sin(\omega x)\, e^{-\omega^2 c^2 t} = c^2 \cdot \left(-\omega^2 \sin(\omega x)\, e^{-\omega^2 c^2 t}\right)$$
$$= c^2 \cdot u_{xx}(x,t),$$

or

$$u_t = c^2 u_{xx},$$

so the function $u(x,t) = \sin(\omega x)\, e^{-\omega^2 c^2 t}$ is indeed a solution of the one-dimensional heat equation.