Question#38948, Math, Trigonometry

I need help with the following proofs.

a.) cos(nθ)=cos((n-2)θ)-2sinθsin((n-1)θ)

b.) (sin(x/2)+sin(x))/(cos(x/2)-cos(x))=cot(x/4)

c.) (sin∝-cos∝+1)/(sin∝+cos∝-1)=(sin∝+1)/cos∝

d.) tan(x/2)=(1-cos(x)+sin(x))/(1+cos(x)+sin(x))

Solution

a.) $\cos(n\theta) = \cos((n-2)\theta)) - 2\sin\theta\sin((n-1)\theta)$

First we rewrite the identity in the form

$$\cos(n\theta) - \cos((n-2)\theta)) = -2\sin\theta\sin((n-1)\theta).$$

Then using

$$\cos A - \cos B = -2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right), \tag{1}$$

we obtain

$$\cos(n\theta) - \cos\big((n-2)\theta\big) = -2\sin\left(\frac{n\theta - n\theta + 2\theta}{2}\right)\sin\left(\frac{2n\theta - 2\theta}{2}\right)$$
$$= -2\sin\theta\sin((n-1)\theta)).$$

Thus, we started with the left side and deduced the right side, so the proof is complete.

b.) $(\sin(x/2) + \sin(x))/(\cos(x/2) - \cos(x)) = \cot(x/4)$

Using

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right),$$

the identity (1) and cancelling we obtain

$$\frac{\sin\frac{x}{2} + \sin x}{\cos\frac{x}{2} - \cos x} = \frac{2\sin\frac{3x}{4}\cos\left(-\frac{x}{4}\right)}{-2\sin\left(-\frac{x}{4}\right)\sin\frac{3x}{4}} = \frac{\cos\frac{x}{4}}{\sin\frac{x}{4}} = \cot\frac{x}{4}.$$

We started with the left side and deduced the right side, so the proof is complete.

c.) $(\sin\alpha - \cos\alpha + 1)/(\sin\alpha + \cos\alpha - 1) = (\sin\alpha + 1)/\cos\alpha$

Because $\sin\alpha + \cos\alpha - 1 \neq 0$ and $\cos\alpha \neq 0$, multiplying the both sides of the identity by their product, we rewrite the identity in the form

$$(\sin\alpha - \cos\alpha + 1)\cos\alpha = (\sin\alpha + \cos\alpha - 1)\cdot(\sin\alpha + 1).$$

Then we start with the left side.

$$(\sin\alpha - \cos\alpha + 1)\cos\alpha = \sin\alpha\cos\alpha + \cos\alpha - \cos^2\alpha.$$

Now we stop and work with the right side.

$$\begin{aligned}(\sin\alpha + \cos\alpha - 1)\cdot(\sin\alpha + 1) &= \sin^2\alpha + \sin\alpha\cos\alpha + \cos\alpha - 1\\ &= \sin\alpha\cos\alpha + \cos\alpha - (1 - \sin^2\alpha)\\ &= \sin\alpha\cos\alpha + \cos\alpha - \cos^2\alpha.\end{aligned}$$

We have used the Pythagorean identity.

We have deduced the same expression from each side, so the proof is complete.

d.) $\tan(x/2)=(1 - \cos(x) + \sin(x))/(1 + \cos(x) + \sin(x))$

We start with the right side. Because $x = 2\cdot\frac{x}{2}$ from double-angle identities

$$\sin 2x = 2\sin x\cos x,\ \cos 2x = 2\cos^2 x - 1 \text{ and } \cos 2x = 1 - 2\sin^2 x$$

we easily get

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}, \qquad 1 - \cos x = 2\sin^2\frac{x}{2}, \qquad 1 + \cos x = 2\cos^2\frac{x}{2}.$$

Using these identities and cancelling we obtain

$$\frac{1 - \cos x + \sin x}{1 + \cos x + \sin x} = \frac{2\sin\frac{x}{2}(\sin\frac{x}{2} + \cos\frac{x}{2})}{2\cos\frac{x}{2}(\cos\frac{x}{2} + \sin\frac{x}{2})} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \tan\frac{x}{2}.$$

We started with the right side and deduced the left one, so the proof is complete.