$\int \tan^{1/3} x \, dx$ Take $\tan x = z$; $\sec^2 x \, dx = dz$ i.e $dx = dz/(1+z^2)$

$= \int z^{1/3} / (1+z^2) \; dz$ Take $t^3 = z$; $dz = 3t^2 dt$
$= \int 3 t^3 / (1+t^6) \; dt$
$= 3 L$ where, $L = \int t^3 / (1+t^6) \; dt$

Now, <u>*by partial fractions*</u>,

$t^3 / (1+t^6) = t^3 / [(1+t^2)(t^4 - t^2 + 1)]$
$= [Ax + B] / [1+t^2] + [Cx + D] / [t^4 - t^2 + 1]$
{ since, (t^2+1) & (t^4-t^2+1) have no real solution }

Now, equivalently,

$t^3 \equiv At^5 + Bt^4 + (C-A)t^3 + (D-B)t^2 + (C+A)t + (B+D)$

Putting t = 0, in the above,
B+D = 0 .... (1)
Putting t = 1,
A + 2C + D = 1 .... (2)
Putting t = - 1,
A + 2C - D = 1 .... (3)
Using (2) & (3), D = 0
Therefore, B = 0 using (1)

A + 2C = 1 .... (4)

So, Now,
$t^3 \equiv At^5 + (C-A)t^3 + (C+A)t$

Putting, t = 2,
13A + 5C = 4 .... (5)

Solving, (4) & (5), C = 3 / 7 & A = 1 / 7

Now,

$L = \int t^3 / (1+t^6) \; dt = 1/7 \int t / (1+t^2) \, dt + 3/7 \int t / (t^4 - t^2 + 1) \; dt$
$= 1 / 14 . \log (1+t^2) + C + 3/7 K$

where C is an arbitary integration constant and $K = \int t / (t^4 - t^2 + 1) \, dt$

$K = \int t / (t^4 - t^2 + 1) \; dt = \int t / [(t^2 - 1/2)^2 + (\sqrt{3} / 2)^2] \; dt$
Take, $t^2 - 1/2 = m$; $2t \, dt = dm$
$K = 1/2 \int dm / [m^2 + (\sqrt{3} / 2)^2] = 1/ \sqrt{3} \; \tan^{-1} (2m / \sqrt{3}) + C_1$
$= 1/ \sqrt{3} \; \tan^{-1} (2 (t^2 - 1/2) / \sqrt{3}) + C_1$

Therefore,
$L = 1 / 14 . \log (1+t^2) + C + \sqrt{3}/7 \; \tan^{-1} (2 (t^2 - 1/2) / \sqrt{3}) + C_1$

Therefore,
$\boldsymbol{\int \tan^{1/3} x \, dx = 3 L}$

$$= 3 \left[\frac{1}{14} \cdot \log(1+t^2) + C + \sqrt{3/7}\, \tan^{-1}\left(2(t^2 - 1/2) / \sqrt{3} \right) + C_1 \right],$$

where $t = \tan^{1/3} x$.