Question:

Obtain the Fourier series for the following periodic function which has a period of 2π. $f(x) = \pi^2 - x^2$ for $-\pi < x < \pi$

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\text{, where}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx,$$

$$f(x) = \pi^2 - x^2;$$

$$f(-x) = \pi^2 - (-x)^2 = \pi^2 - x^2 = f(x);$$

Function is symmetrical.

Then:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) dx, \text{ where}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx,$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} dx = \left.\frac{\pi^2 x - \frac{x^3}{3}}{\pi}\right|_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi^2 - x^2)\cos nx\, dx =$$

$$= \left.\frac{2\left(-\frac{2x\cos(nx)}{n^2} - \frac{(n^2x^2-2)\sin(nx)}{n^3} + \frac{\pi^2\sin(nx)}{n}\right)}{\pi}\right|_0^{\pi} =$$

$$= \frac{2(2\sin(\pi n) - 2\pi n\cos(\pi n))}{\pi n^3}$$

$$f(x) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2(2\sin(\pi n) - 2\pi n\cos(\pi n))}{\pi n^3}\cos nx,$$

Answer:

$$f(x) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2(2\sin(\pi n) - 2\pi n\cos(\pi n))}{\pi n^3}\cos nx,$$