Answer on Question#38806 - Math - Differential Calculus

Question: Solve the following Ordinary Differential Equation:

$$y'' + 4y = 2\cos x \cos 3x$$

Solution. The solution of this equation will be the sum of the complementary solution and the particular solution.

We first find the **complementary solution** by solving the corresponding homogenous equation:

$$y'' + 4y = 0.$$

Assume the solution will be proportional to $e^{\lambda x}$ for some constant λ and substitute $y(x) = e^{\lambda x}$ into the differential equation:

$$(e^{\lambda x})'' + 4e^{\lambda x} = 0$$

$$\lambda^2 e^{\lambda x} + 4e^{\lambda x} = 0$$

Factor out $e^{\lambda x}$:

$$(\lambda^2 + 4)e^{\lambda x} = 0.$$

Since $e^{\lambda x} \neq 0$ for any finite λ, the zeros must come from the polynomial:

$$\lambda^2 + 4 = 0.$$

From this, we can find λ:

$$\lambda_1 = 2i, \lambda_2 = -2i.$$

These roots give us two complementary solutions:

$$y_1(x) = c_1 e^{2ix}, \qquad y_2(x) = c_2 e^{-2ix},$$

Where c_1 and c_2 are arbitrary constants.

Adding these solutions, we obtain the complementary solution of our equation:

$$y_c(x) = y_1(x) + y_2(x) = c_1 e^{2ix} + c_2 e^{-2ix}.$$

This solution can be written in a different way using Euler's identity $e^{a+ib} = e^a (\cos b + i \sin b)$:

$$y_c(x) = c_1 (\cos 2x + i \sin 2x) + c_2 (\cos 2x - i \sin 2x).$$

Regrouping the terms, we get

$$y_c(x) = (c_1 + c_2)\cos 2x + i(c_1 - c_2)\sin 2x.$$

Since c_1 and c_2 are arbitrary constants, we can redefine $c_1 + c_2$ as c_1 and $i(c_1 - c_2)$ as c_2:

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x.$$

Let us now find the **particular solution** by the method of undetermined coefficients.

To do this, note that

$$2\cos x \cos 3x = 2 * \frac{1}{2} * \cos(3x - x) + \cos(3x + x) = \cos 2x + \cos 4x,$$

so we will be solving the equation

$$y'' + 4y = \cos 2x + \cos 4x.$$

The particular solution of will be the sum of particular solutions to two equations:

$$y'' + 4y = \cos 2x,$$
$$y'' + 4y = \cos 4x.$$

These solutions are of the form

$$y_{p1}(x) = x(a_1 \cos 2x + a_2 \sin 2x),$$

$$y_{p2}(x) = a_3 \cos 2x + a_4 \sin 2x,$$

where y_{p1} has been multiplied by x to account for $\cos 2x$ in the complementary solution.

Summing up y_{p1} and y_{p2}, we obtain the particular solution y_p:

$$y_p(x) = y_{p1}(x) + y_{p2}(x) = x(a_1 \cos 2x + a_2 \sin 2x) + a_3 \cos 2x + a_4 \sin 2x.$$

To solve this for unknown constants a_1, a_2, a_3, a_4, substitute y_p into the initial equation:

$$y_p'' + 4y_p = \cos 2x + \cos 4x$$

and simplify:

$$4a_3 \cos 2x - 12a_2 \cos 4x - 4a_1 \sin 2x - 12a_4 \sin 4x = \cos 2x + \cos 4x.$$

Equate the coefficients of $\cos 2x$ on both sides of the equation:

$$4a_3 = 1$$

the coefficients of $\cos 4x$:

$$-12a_2 = 1,$$

the coefficients of $\sin 2x$:

$$-4a_1 = 0,$$

and the coefficients of $\sin 4x$:

$$-12a_4 = 0.$$

Solving the system, we get

$$a_1 = 0, \qquad a_2 = -\frac{1}{12}, \qquad a_3 = \frac{1}{4}, \qquad a_4 = 0.$$

Substitute these values into $y_p(x)$:

$$y_p(x) = -\frac{1}{12}\cos 4x + \frac{1}{4}x \sin 2x.$$

Thus, the solution of our equation is

$$y(x) = y_c(x) + y_p(x) = -\frac{1}{12}\cos 4x + \frac{1}{4}x \sin 2x + c_1 \cos 2x + c_2 \sin 2x.$$

Answer.

$$y(x) = y_c(x) + y_p(x) = -\frac{1}{12}\cos 4x + \frac{1}{4}x \sin 2x + c_1 \cos 2x + c_2 \sin 2x.$$