Question #38805, Math, Geometry

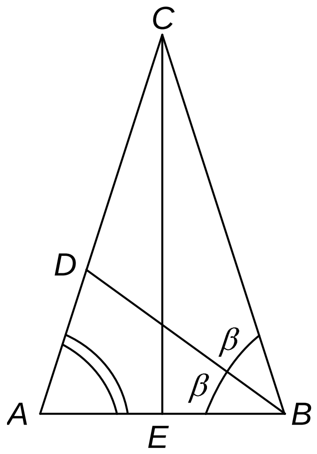
BD BIsects ABC

IF AB=6. BC= 14. AC=14

Find BD

Solution

If we denote the $\angle ABD$ by β then $\angle ABC = 2\beta$. Because $BC = AC$, the triangle ABC is an isosceles one, so we have $\angle ABC = \angle CAB = 2\beta$.



The sine rule states that the sides of a triangle are proportional to the sines of the opposite angles, so from the triangle ABD we obtain the equation (see the Figure):

$$\frac{BD}{\sin\angle CAB} = \frac{AB}{\sin\angle BDA}.$$

The sum of the angles of a triangle is equal 180°, so

$$\angle BDA = 180° - \angle CAB - \angle ABD = 180° - 3\beta.$$

Thus by substituting the angles expression into the above equation and simplifying we obtain

$$BD = AB\frac{\sin 2\beta}{\sin 3\beta}. \quad (1)$$

Two of the basic compound angle formula is

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha, \quad (2)$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\beta\sin\alpha. \quad (3)$$

By substituting β for α into the identity (2) first we get

$$\sin 2\beta = 2\sin\beta\cos\beta. \quad (4)$$

By substituting 2β for α into the formula (2), using the identity (4) and simplifying the result we obtain

$$\sin 3\beta = \sin\beta(\cos 2\beta + \cos^2\beta). \qquad (5)$$

By putting the equalities (4), (5) into the equation (1) and cancelling the fraction we get

$$BD = AB\frac{2\cos\beta}{\cos 2\beta + 2\cos^2\beta}. \qquad (6)$$

Since the triangle *ABC* is an isosceles, then the altitude *CE* is the median of the triangle, that is

$$EB = AE = 0.5AB = 3.$$

We find from the right triangle *EBC*

$$\cos 2\beta = \frac{EB}{CB} = \frac{3}{14}.$$

Putting $\alpha = \beta$ in the (3) with Pythagorean identity gives

$$\cos 2\beta = 2\cos^2\beta - 1.$$

Because the angle β is acute we have

$$\cos\beta = \sqrt{\frac{1 + \cos 2\beta}{2}} = \frac{1}{2}\sqrt{\frac{17}{7}}.$$

Finally, by substituting the two obtained values and $AB = 6$ into the equality (6) we find

$$BD = 6\frac{2 \cdot 0.5}{\frac{17}{14} + \frac{3}{14}}\sqrt{\frac{17}{7}} = \frac{3}{5}\sqrt{119} \approx 6.5452.$$

Answer

$\frac{3}{5}\sqrt{119}$