Question #38685, Math, Calculus

Is the force field defined by F =(3x^2yz-3y) i+(x^3z-3x) ˆj+(x^3y+2)kˆ conservative? If so, find the scalar potential associated with F.

Solution

The domain of the given force field defined by **F** = $(3x^2yz - 3y)$**i** + $(x^3z - 3x)$**j** + $(x^3y + 2)$**k** is all the space that is a simply connected domain, so it is sufficient to verify that curl **F** = **0**, in order to prove that the field is conservative.

Because

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x\hat{}2yz - 3y & x\hat{}3z - 3x & x\hat{}3y + 2 \end{vmatrix} =$$

$$= (x^3 - x^3)\mathbf{i} + (3x^2y - 3x^2y)\mathbf{j} + (3x^2z - 3x^2z)\mathbf{k} = \mathbf{0},$$

we have curl **F** = **0**, hence the field is conservative.

The conservative field **F** can be expressed as grad V where V is a scalar field to be determined (the scalar potential associated with **F**).

We can attempt to express **F** as grad V where V is a scalar in *x, y, z.*

If $V = V(x, y, z)$

$$\text{grad } V = \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}$$

and we have **F** = $(3x^2yz - 3y)$**i** + $(x^3z - 3x)$**j** + $(x^3y + 2)$**k** then

$$\frac{\partial V}{\partial x} = 3x^2yz - 3y, \qquad (1)$$

$$\frac{\partial V}{\partial y} = x^3z - 3x, \qquad (2)$$

$$\frac{\partial V}{\partial z} = x^3y + 2. \qquad (3)$$

By integrating the equality (1) with respect to x, (2) with respect to y and (3) with respect to z we obtain accordingly

$$V = x^3yz - 3xy + f(y, z), \qquad (4)$$

$$V = x^3yz - 3xy + g(x, z), \qquad (5)$$

$$V = x^3yz + 2z + h(x, y). \qquad (6)$$

From equalities (4) and (5) we have

$$f(y, z) = g(x, z) = f_1(z).$$

From the other pair of equalities (5) and (6) the equation follows

$$f_1(z) - 3xy = 2z + h(x, y).$$

This is satisfied if $h(x, y) = -3xy + C$ and $f_1(z) = 2z + C$, where C is an arbitrary constant, hence

$$V = x^3yz - 3xy + 2z + C$$

Answer

The scalar potential associated with **F**: $V = x^3yz - 3xy + 2z + C$