Answer on question 38000 – Math – Calculus

Find the local maximum and minimum values and saddle point(s) of the function f(x,y)=x^3-12xy+8y^3.

Solution

According to the sufficient condition of extremum we should find the point which satisfies the following condition

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}$$

or

$$\begin{cases} 3x^2 - 12y = 0 \\ -12x + 24y^2 = 0 \end{cases}$$

We solve this system of equations by substitution method. From the second equation we find x:

$$x = 2y^2 \qquad (*)$$

And substitute it into the first equation

$$3(2y^2)^2 - 12y = 0$$

$$12y^4 - 12y = 0$$

$$12y(y^3 - 1) = 0$$

$$y = 0 \text{ or } y = 1$$

To find x we should substitute it into (*)

$$x = 0 \text{ or } x = 2$$

Therefor we obtain two point: (0; 0) and (2; 1).

Now we should find the second partial derivatives of our function.

$$\frac{\partial^2 f}{\partial x^2} = 6x, \qquad \frac{\partial^2 f}{\partial y^2} = 48y, \qquad \frac{\partial^2 f}{\partial x \partial y} = -1.$$

We know that the function $f(x, y)$ has maximum at the point $(x_0; y_0)$ if

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0;y_0)} * \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0;y_0)} - \left(\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x_0;y_0)}\right)^2 > 0 \text{ and } \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0;y_0)} < 0;$$

The function has minimum at the point $(x_0; y_0)$ if

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0;y_0)} * \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0;y_0)} - \left(\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x_0;y_0)}\right)^2 > 0 \text{ and } \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0;y_0)} > 0;$$

Using this explore our points on extremum

(0; 0):

$$(6x * 48y - 12^2)|_{(0;0)} = -144 < 0$$

Therefore, (0; 0) is a saddle point.

(2; 1):

$$(6x * 48y - 12^2)|_{(2;1)} = 432 > 0, \qquad \left.\frac{\partial^2 f}{\partial x^2}\right|_{(2;1)} = 6 * 1 = 6 > 0$$

Therefore, (2; 1) is a minimum.

Answer: (0; 0) is a saddle point, (2; 1) is a minimum.