Answer on Question #37999 – Math - Calculus

Question: use the method of Lagrange multipliers to find the local extreme values of the function f subject to the constraint: we introduce the multiplier λ and solve the system, $\nabla f = \lambda \nabla g, g = 0$

1) $f(x, y) = xy$ on the ellipse $x^2 + 4y^2 = 1$.
2) $f(x, y) = 2x - y + 6$ on the circle $x^2 + y^2 = 1$.

Solution: we introduce the Lagrange function $L = f - \lambda \cdot g$ and look for the optimum of this function of the variables x, y, λ.

1) $L = xy - \lambda \cdot (x^2 + 4y^2 - 1)$

$$\begin{cases} \frac{\partial L}{\partial x} = y - 2\lambda x = 0, \\ \frac{\partial L}{\partial y} = x - 8\lambda y = 0 \\ \frac{\partial L}{\partial \lambda} = 1 - x^2 - 4y^2 = 0 \end{cases}$$

From first two equations we obtain the value of λ:

$$y = 2\lambda x, x = 8\lambda y \ \rightarrow y(1 - 16\lambda^2) = 0.$$

Last equation has two solutions:

1) $y = 0$ ($y = 2\lambda x \ \rightarrow x = 0$) – this solution doesn't satisfy the last condition $x^2 + 4y^2 = 1$ and should be rejected.

2) $\lambda = \pm\frac{1}{4}$. From this condition we obtain constraint between y and x: $y = \pm\frac{x}{2}$ and from last equation we find the critical points:

$$x^2 + 4y^2 = 1 \rightarrow 2x^2 = 1 \rightarrow x = \pm\frac{1}{\sqrt{2}}$$

We have four critical points: $\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$.

Extreme values of the initial function are:

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{2}} = \frac{1}{4} - \text{local maximum;}$$
$$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = -\frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{2}} = -\frac{1}{4} - \text{local minimum;}$$

Answer:

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{2}} = \frac{1}{4} - \text{local maximum;}$$
$$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = -\frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{2}} = -\frac{1}{4} - \text{local minimum;}$$

2) $L = 2x - y + 6 - \lambda(x^2 + y^2 - 1)$

$$\frac{\partial L}{\partial x} = 2 - 2\lambda x = 0, \ \ \frac{\partial L}{\partial y} = -1 - 2\lambda y = 0, \ \ \frac{\partial L}{\partial \lambda} = 1 - x^2 - y^2 = 0.$$

From first two equations we obtain that $x = \frac{1}{\lambda}, y = -\frac{1}{2\lambda}$. From third equation we get possible values of λ: $x^2 + y^2 = 1 \ \rightarrow \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} = 1 \ \rightarrow \lambda = \pm\frac{\sqrt{5}}{2}$. We have now two critical points:

$$\left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right), \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

The extreme values of the initial function are respectively

$$f\left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) = \frac{4}{\sqrt{5}} + \frac{1}{\sqrt{5}} + 6 = 6 + \sqrt{5} - \text{local maximum;}$$

$$f\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) = -\frac{4}{\sqrt{5}} - \frac{1}{\sqrt{5}} + 6 = 6 - \sqrt{5} \ - \ \text{local minimum.}$$

Answer:

$$f\left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) = 6 + \sqrt{5} - \text{local maximum;}$$

$$f\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) = 6 - \sqrt{5} \ - \ \text{local minimum.}$$