Answer on question 37890 – Math – Differential equations

Solve

$$y" - 5y' - 6y = e^\wedge(-t)$$

where $y'(0) = 1$ and $y(0) = 1$ using LaPlace Transforms.

Solution

This problem has an inhomogeneous term. In the direct approach one solves for the homogeneous solution and the particular solution separately. For this problem the particular solution can be determined using variation of parameters or the method of undetermined coefficients. Using the Laplace transform technique we can solve for the homogeneous and particular solutions at the same time.

Let Y(s) be the Laplace transform of y(t). Taking the Laplace transform of the differential equation we have:

$$L[y" - 5y' - 6y] = L[e^\wedge(-t)\,]$$

Where

$$L\left[f^{(n)}\right] = s^n F(s) - s^{n-1} f(0) - \cdots f^{(n-1)}(0).$$

The Laplace transform of the LHS $L[y" - 5y' - 6y]$ is

$$L[y" - 5y' - 6y] = L[y''] - 5L[y'] - 6L[y] = s^\wedge 2\, Y(s) - sy(0) - y'(0) - 5sY(s) + 5y(0) - 6Y(s)$$

The Laplace transform of the RHS is

$$L[e^{-t}] = \frac{1}{s+1}$$

Equating the LHS and RHS and using the fact that $y'(0) = 1$ and $y(0) = 1$ we obtain

$$s^2 Y(s) - s - 1 - 5sY(s) + 5 - 6Y(s) = \frac{1}{s+1}$$

$$(s^2 - 5s - 6)Y(s) - s + 4 = \frac{1}{s+1}$$

$$(s^2 - 5s - 6)Y(s) = \frac{1}{s+1} + s - 4 = \frac{1 + (s-4)(s+1)}{s+1} = \frac{s^2 - 3s - 3}{s+1}$$

$$Y(s) = \frac{s^2 - 3s - 3}{(s+1)(s^2 - 5s - 6)} = \frac{s^2 - 3s - 3}{(s+1)^2(s-6)}$$

Using the method of partial fractions it can be shown that

$$Y(s) = \frac{34}{49(s+1)} - \frac{1}{7(s+1)^2} + \frac{15}{49(s-6)}$$

Using the fact that $\frac{1}{s+1}$ is e^{-t}, $\frac{1}{s-6}$ is e^{6t} and that the inverse of $\frac{1}{(s+1)^2}$ is te^{-t}, it follows that

$$y(t) = \frac{34}{49} e^{-t} - \frac{1}{7} te^{-t} + \frac{15}{49} e^{6t}.$$

Answer:

$$y(t) = \frac{34}{49} e^{-t} - \frac{1}{7} te^{-t} + \frac{15}{49} e^{6t}.$$