Question #37829, Math, Calculus

Use a Riemann Sum approximation using midpoints for the following definite integrals using the indicated number of subintervals. Then find the exact area, and compute the error in your approximation.

-definite integral

a=0, b=3.14 (sin(3x))dx; n=12

Solution

Let a continuous function $y = f(x)$ be given on the interval [a, b]. We divide the interval [a, b] into n subintervals by points of division:

$$a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b,$$

so that

$$x_0 < x_1 < x_2 < \ldots < x_n,$$

and put

$$x_1 - x_0 = \Delta x_1, x_2 - x_1 = \Delta x_2, \ldots, x_n - x_{n-1} = \Delta x_n.$$

In each of the intervals [x_0, x_1], [x_1, x_2], …, [x_{n-1}, x_n] we take a point and denote them by t_1, t_2, …, t_n:

$$x_0 < t_1 < x_1, x_1 < t_2 < x_2, \ldots, x_{n-1} < t_n < x_n.$$

At each of these points we find the value of the function $f(t_1)$, $f(t_2)$, …, $f(t_n)$. Then we form a sum:

$$s_n = f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + \ldots + f(t_n)\Delta x_n. \quad (1)$$

This sum is called the Riemann sum of the function $f(x)$ on the interval [a, b].

We divide the interval [0, 3.14] into n = 12 equal subintervals. The length Δx of each subinterval is Δx = (3.14-0)/12= 0.2617; this number is the subinterval (partition unit). The division points have coordinates:

x_0 = 0, x_1 = 0.2617, x_2 = 0.5234, x_3 = 0.7851, x_4 = 1.0468, x_5 = 1.3085, x_6 = 1.5702, x_7 = 1.8319, x_8 = 2.0936, x_9 = 2.3553, x_{10} = 2.6170, x_{11} = 2.8787, x_{12} = 3.1404.

As points t_k we take the midpoints of each subinterval:

t_1 = 0,13085, t_2 = 0,39255, t_3 = 0,65425, t_4 = 0,91595, t_5 = 1,17765, t_6 = 1,43935, t_7 = 1,70105, t_8 = 1,96275, t_9 = 2,22445, t_{10} = 2,48615, t_{11} = 2,74785, t_{12} = 3,00955.

The values of the function $f(t_1)$, $f(t_2)$, …, $f(t_{12})$ at each of these points and their sum are given in the table.

k	t_k	$f(t_k)$
1	0.13085	0.382545695
2	0.39255	0.923708287
3	0.65425	0.924164531
4	0.91595	0.383647359
5	1.17765	-0.381443487
6	1.43935	-0.923250728
7	1.70105	-0.924619461
8	1.96275	-0.384748477
9	2.22445	0.380340736
10	2.48615	0.922791857
11	2.74785	0.925073076
12	3.00955	0.385849048
		2.614058435

From formula (1) we find by substituting $\Delta x_k = \Delta x = 0.2617$

$$S_{12} = f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + \ldots + f(t_{12})\Delta x_{12} = (f(t_1) + f(t_2) + \ldots + f(t_{12}))\Delta x = 2.6141 \cdot 0.2617 = 0.68411.$$

The exact area under the graph of $f(x) = \sin(3x)$ over the interval [0, 3.14] is equal to the definite integral

$$S = \int_0^{3.14} \sin 3x dx = \left[-\frac{\cos 3x}{3}\right]_0^{3.14} = -\frac{\cos 9.42 - 1}{3} = 0.66667.$$

Thus the error in the Riemann Sum approximation is computed as

$$|0.68411 - 0.66667| = 0.01744.$$

Answers:

the Riemann Sum approximation = 0.68411;

the exact area = 0.66667;

the error = 0.01744.