Question #37828, Math, Calculus

Use a Riemann Sum approximation using midpoints for the following definite integrals using the indicated number of subintervals. Then find the exact area, and compute the error in your approximation.

- definite integral:

a=0, b=2 (x^3-3x+3)dx;n=8

Solution

Let a continuous function $y = f(x)$ be given on the interval [a, b]. We divide the interval [a, b] into n subintervals by points of division:

$$a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b,$$

so that

$$x_0 < x_1 < x_2 < \ldots < x_n,$$

and put

$$x_1 - x_0 = \Delta x_1, x_2 - x_1 = \Delta x_2, \ldots, x_n - x_{n-1} = \Delta x_n.$$

In each of the intervals $[x_0, x_1]$, $[x_1, x_2]$, …, $[x_{n-1}, x_n]$ we take a point and denote them by t_1, t_2, …, t_n:

$$x_0 < t_1 < x_1, x_1 < t_2 < x_2, \ldots, x_{n-1} < t_n < x_n.$$

At each of these points we find the value of the function $f(t_1)$, $f(t_2)$, …, $f(t_n)$. Then we form a sum:

$$s_n = f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + \ldots + f(t_n)\Delta x_n. \quad (1)$$

This sum is called the Riemann sum of the function $f(x)$ on the interval [a, b].

We divide the interval [0, 2] into n = 8 equal subintervals. The length Δx of each subinterval is Δx = (2 – 0)/8 = 0.25; this number is the subinterval (partition unit). The division points have coordinates:

$x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$, $x_5 = 1.25$, $x_6 = 1.5$, $x_7 = 1.75$, $x_8 = 2$.

As points t_k we take the midpoints of each subinterval:

$t_1 = 0.125$, $t_2 = 0.375$, $t_3 = 0.625$, $t_4 = 0.875$, $t_5 = 1.125$, $t_6 = 1.375$, $t_7 = 1.625$, $t_8 = 1.875$.

The values of the function $f(t_1)$, $f(t_2)$, …, $f(t_n)$ at each of these points and their sum are given in the table.

k	t_k	$f(t_k)$
1	0.125	2.626953
2	0.375	1.927734
3	0.625	1.369141
4	0.875	1.044922
5	1.125	1.048828
6	1.375	1.474609
7	1.625	2.416016
8	1.875	3.966797
		15.875

From formula (1) we find by substituting $\Delta x_k = \Delta$x = 0.25

$$S_8 = f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + \ldots + f(t_8)\Delta x_8 = (f(t_1) + f(t_2) + \ldots + f(t_8))\Delta x = 15.875 \cdot 0.25 = 3.96875.$$

The exact area under the graph of $f(x) = x^3 - 3x + 3$ over the interval [0, 2] is equal to the definite integral

$$S = \int_0^2 (x^3 - 3x + 3)dx = \left[\frac{x^4}{4} - 3\frac{x^2}{2} + 3x\right]_0^2 = \frac{2^4}{4} - 3\frac{2^2}{2} + 3 \cdot 2 - 0 = 4 - 6 + 6 = 4.$$

Thus the error in the Riemann Sum approximation is computed as

$$|3.96875 - 4| = 0.03125.$$

Answers:

the Riemann Sum approximation = 3.96875;

the exact area = 4;

the error = 0.03125.