Answer on Question 37681 – Math - Real Analysis

If a>0 and b >0 Show that

lim n->infinity squareroot((n+a)(n+b)) -n = (a+b)/2

Solution

1. The sequence 1/n converges to 0 because for given ε > 0 we can choose *N* such that *N* > 1/ε. Then for all *n* > *N*, |1/*n*| < ε, so

$$\lim_{n\to\infty}\frac{1}{n}=0. \qquad (1)$$

The sequence $\frac{1}{\sqrt{n}}$ also converges to 0 because for given $\varepsilon > 0$ we can choose *N* such that *N* > $1/\varepsilon^2$. Then for all *n* > *N*, $\left|\frac{1}{\sqrt{n}}\right| < \varepsilon$, so

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0. \qquad (2)$$

2. If a>0 and b >0 then we have two inequalities

$$\sqrt{1+\frac{a}{n}}\le 1+\frac{\sqrt{a}}{\sqrt{n}}\ and\ \sqrt{1+\frac{b}{n}}\le 1+\frac{\sqrt{b}}{\sqrt{n}}.$$

Hence,

$$1\le\sqrt{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)}\le\left(1+\frac{\sqrt{a}}{\sqrt{n}}\right)\left(1+\frac{\sqrt{b}}{\sqrt{n}}\right).$$

Using (2) and the limit laws for sequences we conclude that

$$\lim_{n\to\infty}\left(1+\frac{\sqrt{a}}{\sqrt{n}}\right)=\lim_{n\to\infty}\left(1+\frac{\sqrt{b}}{\sqrt{n}}\right)=1.$$

Hence, by the Squeeze Theorem we can infer

$$\lim_{n\to\infty}\sqrt{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)}=1. \qquad (3)$$

3. We rationalize the numerator, multiplying by the sum of square roots, and introducing this sum as a denominator, giving

$$\sqrt{(n+a)(n+b)}-n\ =\ \frac{(a+b)n+ab}{\sqrt{(n+a)(n+b)}+n}.$$

This expression is an "$\frac{\infty}{\infty}$" type, and we divide the numerator and denominator by *n* to obtain

$$\sqrt{(n+a)(n+b)}-n\ =\ \frac{(a+b)+\frac{ab}{n}}{\sqrt{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)}+1}\to\frac{a+b}{2}\ as\ n\to\infty,$$

since using (1) and (3) we can conclude by the limit laws for sequences that the numerator tends to *a*+*b* and the denominator tends to 2 as $n\to\infty$.