Answer on Question#37642 - Math - Algebra

Use cylindrical coordinates to find the volume inside the cylinder $x^2 + y^2 = 1$ above the plane $z = 0$, and below the surface $z = 2 - y^2$

Solution:

A cylindrical coordinate system is a three-dimensional coordinate system that specifies point positions by the distance from a chosen reference axis, the direction from the axis relative to a chosen reference direction and the distance from a chosen reference plane perpendicular to the axis. The latter distance is given as a positive or negative number depending on which side of the reference plane faces the point. Cylindrical coordinates are just polar coordinates on the plane z = 0 together with the vertical coordinate z. In the cylindrical coordinate system, a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where:

- **r** and **θ** are polar coordinates of the projection of P onto the $xy\text{–}plane$.
- **z** is the directed distance from the $xy - plane$ to P.

To convert from rectangular to cylindrical coordinates, we use:

$$r^2 = x^2 + y^2$$

$$tan\theta = \frac{y}{x}$$

$$t = z$$

The conversion formulas for cylindrical coordinates are:

$$x = rcos\theta$$

$$y = rsin\theta$$

$$z = z$$

where $0 \le r < \infty$, $0 \le \theta \le 2\pi$, $-\infty < z < \infty$ and we have to note that $r = \sqrt{x^2 + y^2}$

$$\theta = tan^{-1}\left(\frac{y}{x}\right)$$

In cylindrical coordinates the volume is bounded by cylinder r = 1 and the surface $z = 2 - y^2$. So we can write $(1, \theta, z)$.

The formula which relates an integral over rectangular coordinates to a corresponding integral over cylindrical coordinates. If a region E in (x, y, z) space is given by cylindrical inequalities $0 \le a \le r \le b, \alpha \le \theta \le \beta (0 \le \beta - \alpha \le 2\pi), z_1 \le z \le z_2$ then we have an inequality of integrals:

$$\int\int_E\int f(x, y, z)dV = \int_\alpha^\beta \int_a^b \int_{z_1}^{z_2} f(rcos\theta, rsin\theta, z)rdzdrd\theta$$

According to our task we can write and our boundaries are:

$$0 \le z \le 2 - y^2$$

In cylindrical coordinates

$$0 \le z \le 2 - r^2 \sin^2 \theta$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq \sqrt{x^2 + y^2} = 1$$

So, we can to calculate the integral:

$$V = \int_0^1 \int_0^{2\pi} \int_0^{2-r^2cos^2\theta} rdzd\theta dr = \int_0^1 dr \int_0^{2\pi} (2 - r^2 \cos^2\theta) rd\theta = \int_0^1 (4\pi r - \pi r^3)dr = \frac{7}{4}\pi$$

Also we can solve this task by solving the equation $z = 2 - y^2$. We can get the z-limits. They are:

$$0 \leq z \leq 2 - y^2$$

Then we solve $x^2 + y^2 = 1$ to get the y-limits. They are:

$y = \pm\sqrt{1 - x^2}$ or we can write $-\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}$

After that we find the x-limits: $-1 \leq x \leq 1$

Here is that in integral form:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1+x^2}} \int_0^{2-y^2} dzdydx \approx 5.498$$

Answer: $V = \frac{7}{4}\pi \approx 5.495.$