Answer on Question #37628 – Math – Discrete Mathematics

Question. Find the smallest equivalence relation on $A = \{1, 2, 3\}$ that contains $(1, 2)$ and $(2, 3)$.

Solution. By definition a relation R on a set A is an arbitrary subset of $A \times A$. A relation R is called *equivalence* if

(1) R is *reflexive*, that is $(x, x) \in R$ for all $x \in A$;

(2) R is *symmetric*, that is if $(x, y) \in R$, then $(y, x) \in R$ for all $x, y \in A$;

(3) R is *transitive*, that is if $(x, y), (y, z) \in R$, then $(x, z) \in R$ as well for all $x, y, z \in A$.

Suppose $R \subset A \times A$ is an equivalence relation on $A = \{1, 2, 3\}$ containing $(1, 2)$ and $(2, 3)$. We claim that then $R = A \times A$.

Indeed, since R is reflexive, $(1, 1)$, $(2, 2)$, and $(3, 3) \in R$.

As R is transitive, and $(1, 2), (2, 3) \in R$, we obtain that $(1, 3) \in R$ as well.

Since R is symmetric, we get that then $(2, 1)$, $(3, 2)$ and $(3, 1) \in R$ as well.

Thus we see that each element $(i, j) \in A \times A$ belongs to R, and so $R = A \times A$.

Thus $R = A \times A$ is a unique equivalence relation on A containing $(1, 2)$ and $(2, 3)$.

Answer. $R = A \times A$.