Answer on Question 37613 - Math – Geometry

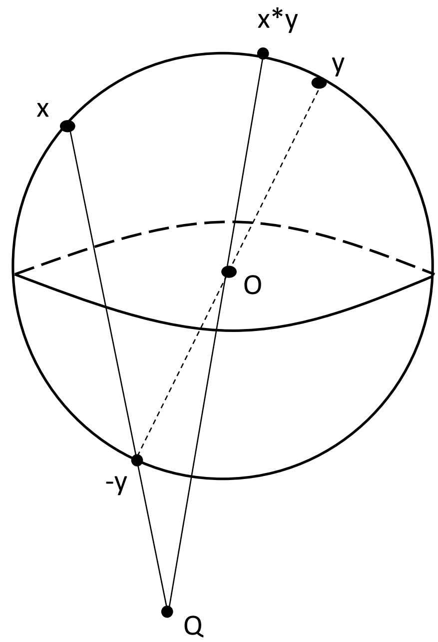
Recall that we can thought of a projective plane as an upper hemisphere. We define the binary operation on a projective plane as follows: Let x and y be any two points belong to the projective plane, think of the projective plane as an upper hemisphere. Then take the antipodal point of y which is in the lower hemispher and identified with y. Draw a line through x and the antipodal point of y. Then draw a line parallel to that line with same length and the origin is the mid point, x*y is then defined as the endpoint of this line (Since the end points are antipodal so they are equal in RP).
My question is
Is it true that x*(y*z)=(x*y)*(x*z) for x,y,z in RP

Solving

Assume that it is true. Then this equation holds for any three points that belong to the upper hemisphere. Let points x, y, z has coordinates $(0,0,1)$, $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ resdpectively. And for simplicity we assume that center of the sphere at the origin of coordinates and radios equals 1.Than equation of the sphere is $x^2 + y^2 + z^2 = 1$.

Let's find for this points x*(y*z) and (x*y)*(x*z).

x*y: -y has coordinates $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$. We should find coordinates of point Q. The point –y devides the segment xQ in the ratio of 2 to 1. Using the formula of dividing the segment in the given ratio we get



$-\frac{1}{\sqrt{3}} = \frac{0+2*x}{3} = \frac{2x}{3}$

$x = -\frac{3}{2\sqrt{3}}$

Similarly we get

$y = -\frac{3}{2\sqrt{3}}, z = -\frac{3+\sqrt{3}}{2\sqrt{3}}$.

Equation of the line QO is

$$\begin{cases} x = \frac{3}{2\sqrt{3}}t \\ y = \frac{3}{2\sqrt{3}}t \\ z = \frac{3+\sqrt{3}}{2\sqrt{3}}t \end{cases}$$

To find the coordinates of the point x*y we should substitute this into equation of the sphere

$$\left(\frac{3}{2\sqrt{3}}t\right)^2 + \left(\frac{3}{2\sqrt{3}}t\right)^2 + \left(\frac{3+\sqrt{3}}{2\sqrt{3}}t\right)^2 = 1$$

$$t = \frac{1}{\sqrt{5+\sqrt{3}}}$$

x*y: $\left(\frac{3}{2\sqrt{15+3\sqrt{3}}}, \frac{3}{2\sqrt{15+3\sqrt{3}}}, \frac{3+\sqrt{3}}{2\sqrt{15+3\sqrt{3}}}\right)$.

x*z: -z has coordinates $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$. Similarly to the previous case we should find coordinates of point Q'.

$-\frac{1}{\sqrt{3}} = \frac{0+2*x}{3} = \frac{2x}{3}, \ x = -\frac{3}{2\sqrt{3}}$

$\frac{1}{\sqrt{3}} = \frac{0+2*y}{3} = \frac{2y}{3}, \ y = \frac{3}{2\sqrt{3}}$

$-\frac{1}{\sqrt{3}} = \frac{1+2z}{3}, \ z = -\frac{3+\sqrt{3}}{2\sqrt{3}}.$

Equation of the line Q'O is

$$\begin{cases} x = \frac{3}{2\sqrt{3}}\mathrm{t} \\ y = -\frac{3}{2\sqrt{3}}\mathrm{t} \\ z = \frac{3+\sqrt{3}}{2\sqrt{3}}\mathrm{t} \end{cases}$$

To find the coordinates of the point x*z we should substitute this into equation of the sphere

$$\left(\frac{3}{2\sqrt{3}}\mathrm{t}\right)^2 + \left(-\frac{3}{2\sqrt{3}}\mathrm{t}\right)^2 + \left(\frac{3+\sqrt{3}}{2\sqrt{3}}\mathrm{t}\right)^2 = 1$$

$$t = \frac{1}{\sqrt{5+\sqrt{3}}}$$

x*z: $\left(\frac{3}{2\sqrt{15+3\sqrt{3}}}, -\frac{3}{2\sqrt{15+3\sqrt{3}}}, \frac{3+\sqrt{3}}{2\sqrt{15+3\sqrt{3}}}\right)$

y*z: -z has coordinates $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$. Similarly to the previous cases we should find coordinates of point Q''.

$-\frac{1}{\sqrt{3}} = \frac{\frac{1}{\sqrt{3}}+2*x}{3} = \frac{1+2\sqrt{3}x}{3}, \ \ x = -\frac{2}{\sqrt{3}}$

$\frac{1}{\sqrt{3}} = \frac{\frac{1}{\sqrt{3}}+2*y}{3} = \frac{1+2\sqrt{3}y}{3}, \ \ y = \frac{1}{\sqrt{3}}$

$-\frac{1}{\sqrt{3}} = \frac{\frac{1}{\sqrt{3}}+2*z}{3} = \frac{1+2\sqrt{3}z}{3}, \ \ z = -\frac{2}{\sqrt{3}}$

Equation of the line Q''O is

$$\begin{cases} x = -\frac{2}{\sqrt{3}}\mathrm{t} \\ y = \frac{1}{\sqrt{3}}\mathrm{t} \\ z = -\frac{2}{\sqrt{3}}\mathrm{t} \end{cases}$$

To find the coordinates of the point x*z we should substitute this into equation of the sphere

$$\left(-\frac{2}{\sqrt{3}}\mathrm{t}\right)^2 + \left(\frac{1}{\sqrt{3}}\mathrm{t}\right)^2 + \left(-\frac{2}{\sqrt{3}}\mathrm{t}\right)^2 = 1$$

$$t = \frac{1}{\sqrt{3}}$$

y*z: $\left(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$

Now we can find x*(y*z).

-y*z is $\left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$

The coordinates of the endpoint K is

$\frac{2}{3} = \frac{0+2*x}{3} = \frac{2x}{3}, \ $ x= 1

$-\frac{1}{3}=\frac{2y}{3},\ y=-\frac{1}{2}$

$-\frac{2}{3}=\frac{1+2z}{3},\ z=-\frac{3}{2}$

Equation of the line KO is

$$\begin{cases} x=-\mathrm{t} \\ y=\frac{1}{2}\mathrm{t} \\ z=\frac{3}{2}\mathrm{t} \end{cases}$$

To find the coordinates of the point x*(y*z) we should substitute this into equation of the sphere

$$(-\mathrm{t})^2+\left(\frac{1}{2}\mathrm{t}\right)^2+\left(\frac{3}{2}\mathrm{t}\right)^2=1$$

$$t=\frac{2}{\sqrt{14}}$$

x*(y*z): $(-\frac{2}{\sqrt{14}},\frac{1}{\sqrt{14}},\frac{3}{\sqrt{14}})$.

It is remain to find (x*y)*(x*z).

-(x*z) has coordinates $(-\frac{3}{2\sqrt{15+3\sqrt{3}}},\frac{3}{2\sqrt{15+3\sqrt{3}}},-\frac{3+\sqrt{3}}{2\sqrt{15+3\sqrt{3}}})$

Let us find the point K'.

$-\frac{3}{2\sqrt{15+3\sqrt{3}}}=\frac{\frac{3}{2\sqrt{15+3\sqrt{3}}}+2x}{3},\ x=-\frac{3}{\sqrt{15+3\sqrt{3}}}$

$\frac{3}{2\sqrt{15+3\sqrt{3}}}=\frac{\frac{3}{2\sqrt{15+3\sqrt{3}}}+2y}{3},\ y=\frac{3}{2\sqrt{15+3\sqrt{3}}}$

$-\frac{3+\sqrt{3}}{2\sqrt{15+3\sqrt{3}}}=\frac{\frac{3+\sqrt{3}}{2\sqrt{15+3\sqrt{3}}}+2z}{3},\ z=\frac{\sqrt{3}+1}{\sqrt{5+\sqrt{3}}}$

Equation of the line K'O is

$$\begin{cases} x=-\frac{3}{\sqrt{15+3\sqrt{3}}}\mathrm{t} \\ y=\frac{3}{2\sqrt{15+3\sqrt{3}}}\mathrm{t} \\ z=\frac{3+\sqrt{3}}{\sqrt{15+3\sqrt{3}}}\mathrm{t} \end{cases}$$

To find the coordinates of the point (x*y)*(x*z) we should substitute this into equation of the sphere

$$\left(-\frac{3}{\sqrt{15+3\sqrt{3}}}\mathrm{t}\right)^2+\left(\frac{3}{2\sqrt{15+3\sqrt{3}}}\mathrm{t}\right)^2+\left(\frac{3+\sqrt{3}}{\sqrt{15+3\sqrt{3}}}\mathrm{t}\right)^2=1$$

$$\frac{9}{15+3\sqrt{3}}\mathrm{t}^2+\frac{9}{4(15+3\sqrt{3})}\mathrm{t}^2+\frac{(3+\sqrt{3})^2}{15+3\sqrt{3}}\mathrm{t}^2=1$$

$$(93+24\sqrt{3})t^2=4(15+3\sqrt{3})$$

$$t=\sqrt{\frac{4(15+3\sqrt{3})}{(93+24\sqrt{3})}}$$

(x*y)*(x*z): $\left(-\frac{6}{\sqrt{(93+24\sqrt{3})}}, \frac{3}{\sqrt{(93+24\sqrt{3})}}, \frac{2(3+\sqrt{3})}{\sqrt{(93+24\sqrt{3})}}\right)$

Now we can see that x*(y*z)≠(x*y)*(x*z).

Answer: it is no true.