Answer on Question #37498 – Math - Integral Calculus

If a function is defined as a function of x by $y = f(x)$ then the arc length is given by:

$$s = \int_a^b \sqrt{1 + (y')^2}\, dx$$

We have $y(x) = \ln \sec x$:

$$y'(x) = (\ln \sec x)' = \frac{1}{\sec x} \cdot (\sec x)' = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x$$

Then take the integral:

$$s = \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/3} \sqrt{\frac{1}{\cos^2 x}} dx = \int_0^{\pi/3} \sqrt{\frac{1}{\cos^2 x}} dx = \left|\cos x > 0 \text{ for } x \in \left[0, \frac{\pi}{3}\right]\right| =$$

$$= \int_0^{\pi/3} \frac{1}{\cos x} dx = \int_0^{\pi/3} \frac{\cos x}{\cos^2 x} dx = \int_0^{\pi/3} \frac{d \sin x}{1 - \sin^2 x}$$

Subsitute $\sin x = t$. Then $0 \to \sin 0 = 0, \frac{\pi}{3} \to \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$:

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{dt}{1 - t^2} = \int_0^{\frac{\sqrt{3}}{2}} \frac{dt}{(1 - t)(1 + t)} = \frac{1}{2}\int_0^{\frac{\sqrt{3}}{2}} \frac{dt}{(t + 1)} - \frac{1}{2}\int_0^{\frac{\sqrt{3}}{2}} \frac{dt}{(t - 1)} = \frac{1}{2}[\ln(t + 1) - \ln(t - 1)]\Big|_0^{\frac{\sqrt{3}}{2}} =$$

$$\frac{1}{2}\ln\frac{t + 1}{t - 1}\Big|_0^{\frac{\sqrt{3}}{2}} = \frac{1}{2}\left[\ln\frac{\frac{\sqrt{3}}{2} + 1}{\frac{\sqrt{3}}{2} - 1} - \ln 1\right] = \frac{1}{2}\ln\left(\frac{\sqrt{3} + 2}{\sqrt{3} - 2}\right) = \frac{1}{2}\ln\left(\frac{(\sqrt{3} + 2)(\sqrt{3} + 2)}{(\sqrt{3} - 2)(\sqrt{3} + 2)}\right) =$$

$$= \frac{1}{2}\ln\left(\frac{(\sqrt{3} + 2)^2}{3 - 2}\right) = \mathbf{\ln(2 + \sqrt{3}) = s}$$

So the length of the curve $y = \log \sec x$ between the points $x = 0$ and $x = \pi/3$ is $\ln(2 + \sqrt{3})$.