Answer on Question #37226 – Math - Differential Calculus

Find the Taylors expansion of f(x) = Sin x at x = 0

Solution.

Taylor's theorem.

Suppose $f(x)$ has derivatives of all orders at $x = b$. The Taylor series of $f(x)$ expanded about $x = b$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!}(x-b)^n$$

Let`s find the Taylor series of $f(x) = \sin x$ expanded about $x = 0$.

First, compute the derivatives and look for a pattern.

$$f'(x) = \cos x \qquad f'(0) = 1$$

$$f''(x) = -\sin x \qquad f''(0) = 0$$

$$f'''(x) = -\cos x \qquad f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cos x \qquad f^{(5)}(0) = 1$$

So we see

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots +$$

The n^{th} term is $\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}$.

To find the interval of convergence, observe that for all , and for all x,

$$\left|f^{(n+1)}(x)\right| \le 1$$

So in Taylor's inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-b|^{n+1}$$

we can take $M = 1$ and $C = \infty$, for each x obtaining $|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$

Now apply the Ration Test

$$\lim_{n\to\infty} \frac{|x|^{n+2}/(n+2)!}{|x|^{n+1}/(n+1)!} = \lim_{n\to\infty} \frac{|x|}{(n+2)!} = 0$$

for all x and we see the radius of convergence is infinite.