Answer on Question#37193 - Math - Other

If we are given an arbitrary $I \times J$ matrix A, there exists $a > 0$ so that the matrix B with entries $B_{ij} = A_{ij} + a$ has only positive entries. Show that any optimal randomized probability vectors for the game with pay-off matrix B are also optimal for the game with pay-off matrix A.

Solution:

Suppose that vectors

$$\overline{x} = (x_1; x_2; \dots; x_I),\ \ \overline{y} = (y_1; y_2; \dots; y_J),$$

are optimal randomized probability ones for the game with pay-off matrix B. Thus (by the definition of an optimal randomized probability vector)

$$\sum_{i=1}^{I} x_i = 1,\ \ \sum_{j=1}^{J} y_j = 1.$$

If vectors $\overline{x}, \overline{y}$ are optimal randomized probability ones then

$$v_B = v_B^{\beta} = \min_{\overline{y}} \max_{\overline{x}} \sum_{i=1}^{I} \sum_{j=1}^{J} B_{ij} x_i y_j = \max_{\overline{x}} \min_{\overline{y}} \sum_{i=1}^{I} \sum_{j=1}^{J} B_{ij} x_i y_j = v_B^{\alpha}$$

where v_B is a value of the game with pay-off matrix B.

Thus we have

$$v_A^{\beta} = \min_{\overline{y}} \max_{\overline{x}} \sum_{i=1}^{I} \sum_{j=1}^{J} A_{ij} x_i y_j = \min_{\overline{y}} \max_{\overline{x}} \sum_{i=1}^{I} \sum_{j=1}^{J} (B_{ij} - a) x_i y_j = \min_{\overline{y}} \max_{\overline{x}} \sum_{i=1}^{I} \sum_{j=1}^{J} B_{ij} x_i y_j -$$

$$- \min_{\overline{y}} \max_{\overline{x}} \sum_{i=1}^{I} \sum_{j=1}^{J} a x_i y_j = v_B - a \min_{\overline{y}} \max_{\overline{x}} \sum_{i=1}^{I} \sum_{j=1}^{J} x_i y_j = v_B - a \min_{\overline{y}} \max_{\overline{x}} \sum_{i=1}^{I} x_i \sum_{j=1}^{J} y_j =$$

$$= v_B - a \min_{\overline{y}} \max_{\overline{x}} (1 \cdot 1) = v_B - a.$$

By analogy

$$v_A^{\alpha} = v_B - a.$$

Because

$$v_A^{\beta} = v_B - a = v_A^{\alpha}$$

thus the game with pay-off matrix A has the value

$$v_A = v_B - a$$

and vectors

$$\overline{x} = (x_1; x_2; \dots; x_I),\ \ \overline{y} = (y_1; y_2; \dots; y_J),$$

are optimal randomized probability ones for the game with pay-off matrix A.