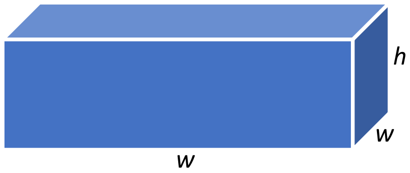
Question: A box with a square base and open top must have a volume of $364500\ \mathrm{cm}^3$. Find the dimensions of the box that minimize the amount of material used.

Solution. Let l be the length, w the width and h the height of the box. Note that since the box has a square base, $l = w$.



The volume of the box is

$$V = l * w * h = w^2 * h.$$

Since we know that $V = 364500$, we can express h as a function of w:

$$h = \frac{V}{w^2} = \frac{364500}{w^2}.$$

Let us now calculate the amount of material used. This corresponds to the surface area:

$$S = w^2 + 4 * w * h.$$

Here we again use the fact that the base is square, as well as the condition that the box has an open top (and so the surface area is equal to the sum of four side areas and *one* base area instead of two).

We now substitute h in the formula for surface area with the expression found previously:

$$S = w^2 + 4 * w * \frac{364500}{w^2} = w^2 + \frac{4 * 364500}{w}.$$

We need to minimize this expression in order to fulfil the "minimum amount of material" condition. As we know, the minimum of a function can be found by means of differentiating:

$$S'(w) = 2w - \frac{4 * 364500}{w^2}$$

and setting the derivative equal to zero:

$$2w - \frac{4 * 364500}{w^2} = 0.$$

This is an equation for finding w:

$$2w = \frac{4 * 364500}{w^2}$$

$$w^3 = 2 * 364500 = 729000,$$

and so

$$w = 90\ (\mathrm{cm}).$$

As the final step, we find h:

$$h = \frac{364500}{w^2} = \frac{364500}{90^2} = \frac{364500}{8100} = 45\ (\mathrm{cm}).$$

Answer. The dimensions of the box are as follows: length 90 cm, width 90 cm, height 45 cm.