The equation $f(t)=\frac{5t}{t^2+3t+2}$ models the bacteria count, in thousands, for a sample of tap water that is left to sit over time, t, in days. The equation $g(t)=\frac{15t}{t^2+9}$ models the bacteria count, in thousands, for a sample of pond water that is also left to sit over several days. In both models, $t>0$. Will the bacteria count for the tap water sample ever exceed the bacteria count for the pond water? Justify your answer.

Solution.

Let's find points of intersections of the functions $f(t)$ and $g(t)$:

$$f(t)=g(t),\ \ t>0$$

$$\frac{5t}{t^2+3t+2}=\frac{15t}{t^2+9}\Bigg|\times\frac{(t^2+3t+2)(t^2+9)}{t}$$

$$5(t^2+9)=15(t^2+3t+2)$$

$$5t^2+45-15t^2-45t-30=0$$

$$-10t^2-45t+15=0$$

$$2t^2+9t-3=0$$

$$t_{1,2}=\frac{-9\pm\sqrt{81+4\cdot 2\cdot 3}}{4}=\frac{-9\pm\sqrt{105}}{4}$$

$$t_1=\frac{1}{4}\left(-9+\sqrt{105}\right),\ \ t_2=\frac{1}{4}\left(-9-\sqrt{105}\right)$$

In both models, $t>0$. Therefore, we have only one solution:

$$t_s=\frac{1}{4}\left(-9+\sqrt{105}\right)\approx 0.312$$

Consider the intervals $t\in(0,t_s)$ and $t\in(t_s,\infty)$. On one of the intervals $f(t)>g(t)$. Let's investigate it:

$t_I=0.1\in(0,t_s)$:

$$f(t_I)=\frac{5\cdot 0.1}{0.1^2+3\cdot 0.1+2}\approx 0.216,\qquad g(t_I)=\frac{15\cdot 0.1}{0.1^2+9}\approx 0.166$$

So

$$f(t_I)>g(t_I)$$

Then

$$f(t)>g(t)\text{ for }t\in\left(0,\frac{1}{4}\left(-9+\sqrt{105}\right)\right)$$

It means that the bacteria count for the tap water will exceed the bacteria count for the pond water, when $t<\frac{1}{4}\left(-9+\sqrt{105}\right)$ days.

Answer:

$$t<\frac{1}{4}\left(-9+\sqrt{105}\right)\text{ days}$$