

Question: The Turtledove Chocolate factory has two chocolate machines. Machine A takes s minutes to fill a case with chocolates, and machine B takes $s + 10$ minutes to fill a case. Working together, the two machines take 15 minutes to fill a case. Approximately how long does each machine take to fill a case?

Solution. We shall use the following general formula:

$$Time * Productivity = Total\ amount\ of\ work\ done,$$

where *Productivity* is the amount of work done during a unit of time.

In our case, “amount of work done” is the volume filled with chocolates, and *Time* is measured in minutes.

Let us begin by finding the amount of work each machine does in one minute. Dividing total amount of work (1 case) by the corresponding amounts of time (s and $s + 10$ minutes), we find that in one minute machines A and B fill $\frac{1}{s}$ part of the case and $\frac{1}{s+10}$ part of the case, respectively. We also note that working together, the two machines fill $\frac{1}{15}$ of the case in one minute.

Let us write this down as a table:

Machine	Time to fill 1 case	Productivity (volume filled in 1 minute)
A	s minutes	$\frac{1}{s}$ of the case
B	$s + 10$ minutes	$\frac{1}{s+10}$ of the case
Both (working together)	15 minutes	$\frac{1}{15} = \frac{1}{s} + \frac{1}{s+10}$

Let us explain the result in the bottom right cell.

On the one hand, we can find the productivity of two machines working together by adding individual productivities, and thus obtain $\frac{1}{s} + \frac{1}{s+10}$.

On the other hand, the productivity of two machines can be found from the initial formula, just as we found individual productivities. Dividing amount of work done (1 case) by time (15 minutes), we obtain $\frac{1}{15}$.

We now have an equation for finding s . Let us solve it:

$$\frac{1}{15} = \frac{1}{s} + \frac{1}{s+10}$$

Bring the right-hand side over a common denominator:

$$\frac{1}{15} = \frac{s+10+s}{s*(s+10)}$$

$$s*(s+10) = 15*(2s+10)$$

$$s^2 + 10s = 30s + 150$$

$$s^2 - 20s - 150 = 0$$

This is a quadratic equation. Using the standard *quadratic formula*, we have

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{20 \pm \sqrt{20^2 + 4 * 150}}{2} = \frac{20 \pm \sqrt{1000}}{2} = \frac{20 \pm 10\sqrt{10}}{2}$$

Therefore,

$$s = 10 \pm 5\sqrt{10}.$$

Note, however, that s must be nonnegative (since it denotes an amount of time). Thus, $10 - 5\sqrt{10} < 0$ is not a valid solution, and we have

$$s = 10 + 5\sqrt{10} \approx 10 + 5 * 3.16 = 25.8.$$

Machine A fills the case with chocolates in approximately 26 minutes. Now for machine B we have

$$s + 10 \approx 35.8,$$

so machine B fills the case in approximately 36 minutes.

Answer. Working alone, machines A and B fill the case with chocolates in approximately 26 minutes and 36 minutes, respectively.