Question #36597, Calculus

Let f : R → R be the function

f(x) = (x – a1)(x – a2) + (x – a2)(x – a3) + (x – a3)(x – a1) with a1, a2, a3 ∈ ℝ. Then f(x) ≥ 0 if and only if :

Solution

Open brackets in the expression of f(x), then

f(x) = $x^2-(a_1+a_2)x+a_1a_2+x^2-(a_2+a_3)x+a_2a_3+x^2-(a_1+a_3)x+a_1a_3$=

$=3x^2-2(a_1+a_2+a_3)x+(a_1a_2+a_2a_3+a_1a_3)$.

Take into account that this parabola opens upward.

Moreover, f(x) ≥ 0 if and only if

$D=4(a_1+a_2+a_3)^2-4*3*(a_1a_2+a_2a_3+a_1a_3)=$

$=4(a_1^2+a_2^2+a_3^2+2a_1a_2+2a_2a_3+2a_1a_3)-12a_1a_2-12a_2a_3-12a_1a_3=$

$=4(a_1^2+a_2^2+a_3^2-a_1a_2-a_2a_3-a_1a_3)=$

$=4*\left(\frac{a_1}{\sqrt{2}}-\frac{a_2}{\sqrt{2}}\right)^2+4*\left(\frac{a_2}{\sqrt{2}}-\frac{a_3}{\sqrt{2}}\right)^2+4*\left(\frac{a_1}{\sqrt{2}}-\frac{a_3}{\sqrt{2}}\right)^2\le 0.$

But we know that the sum of squares is non-negative.

Therefore, the only possible variant is

$\frac{a_1}{\sqrt{2}}-\frac{a_2}{\sqrt{2}}=0, \frac{a_2}{\sqrt{2}}-\frac{a_3}{\sqrt{2}}=0, \frac{a_1}{\sqrt{2}}-\frac{a_3}{\sqrt{2}}=0$, from where we conclude

$a_1=a_2=a_3$.

Answer: $a_1=a_2=a_3$