Let a be a fixed element of a group G. Show that $H = \{x \in G : xa = ax\}$ is a subgroup of G.

Solution.

Check that $\forall\, x, y \in H : x^{-1} \in H, x \cdot y \in H$:

Let $x \in H$.

$$xa = ax \Rightarrow x^{-1}(xa) = x^{-1}(ax) \Rightarrow a = x^{-1}ax \Rightarrow ax^{-1} = (x^{-1}ax)x^{-1} \Rightarrow$$

$$\Rightarrow ax^{-1} = x^{-1}a \Rightarrow x^{-1} \in H\ ;$$

Let $x, y \in H$.

$$ya = ay \Rightarrow xya = x(ay)\ ;$$

$$xa = ax \Rightarrow x(ay) = (ax)y\ ;$$

Hence:

$$xya = axy \Rightarrow xy \in H\ .$$

Thus,

1) $\forall x \in H : x^{-1} \in H$;
2) $\forall x, y \in H : xy \in H$;

Besides:

$\forall x \in H : x \cdot x^{-1} \in H \Rightarrow e \in H$ (e - neutral element) ;

$\forall x, y, z \in H : x, y, z \in G \Rightarrow x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associativity);

So, H is a subgroup of G.