Let W_1 and W_2 be subspaces of a vector space V and define $W_1 + W_2$:= {u+v : u ∈ W_1, v ∈ W_2}. Prove that $span$ (W_1 U W_2) = $W_1 + W_2$.

Solution

Let W_1 and W_2 be subspaces of a vector space V and define $W_1 + W_2$:= {u+v : u ∈ W_1, v ∈ W_2}. In other words, $W_1 + W_2$ is the collection of all vectors you can get by adding an element of W_1 to an element of W_2.

Prove that $span$ (W1 U W_2) = $W_1 + W_2$ (in other words $W_1 + W_2$ is the smallest subspace containing both W1 and W_2).

To see $W_1 + W_2$ is a subspace, check closure under addition and under multiplication by scalars. Let $u + v$ and $\acute{u} + \acute{v}$ be any elements of $W_1 + W_2$, where $u, \acute{u} \in W_1$ and $v, \acute{v} \in W_2$, and let a be any scalar. Then, since W_1 and W_2 are closed under addition and under multiplication by scalars,

$$(u + v) + (\acute{u} + \acute{v}) = (u + \acute{u}) + (v + \acute{v}) \in W_1 + W_2,$$

$$a(u + v) = au + av \in W_1 + W_2.$$

Also, $W_1 \subseteq W_1 + W_2$, since every u ∈ W_1 can be written $u + 0 \in W_1 + W_2$. For the same reason, $W_2 \subseteq W_1 + W_2$. We have shown $W_1 + W_2$ is a subspace containing both W_1 and W_2.

Clearly every element $u + v \in W_1 + W_2$ is in $span(W_1 + W_2)$ so $W_1 + W_2 \subseteq span(W_1 \cup W_2)$.

To show $W_1 + W_2 = span(W_1 \cup W_2)$., it remains only to show that $span(W_1 \cup W_2) \subseteq W_1 + W_2$. But this must be true, because we have shown $W_1 + W_2$ is a subspace containing $W_1 \cup W_2$, and $span(W_1 \cup W_2)$ is the smallest such subspace.

So we have shown that $span$ (W1 U W_2) = $W_1 + W_2$.