Use Farka's Lemma directly to prove that, if p^* is finite, then PS has a feasible solution.

Solution.

Farkas' Lemma: precisely one of the following is true:

a. there is $x \geq 0$ such that $Ax = b$;
b. there is y such that $A^T y \geq 0$ and $b^T y \leq 0$.

So consider the system of inequalities given in block-matrix form by

$$\begin{bmatrix} -A^T & c \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} r \\ \alpha \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -b^T & p^* \end{bmatrix} \begin{bmatrix} r \\ \alpha \end{bmatrix} < 0$$

Here r is a column vector and α is a real number.

By Farkas' Lemma, there must be $\hat{x} \geq 0$ and $\beta \geq 0$ such that $A\hat{x} = b$ and $c^T \hat{x} = p^* - \beta \leq p^*$. It follows that $\hat{x}$ is optimal (feasible solution where the objective function reaches its maximum or minimum) for PS and $c^T \hat{x} = p^*$.