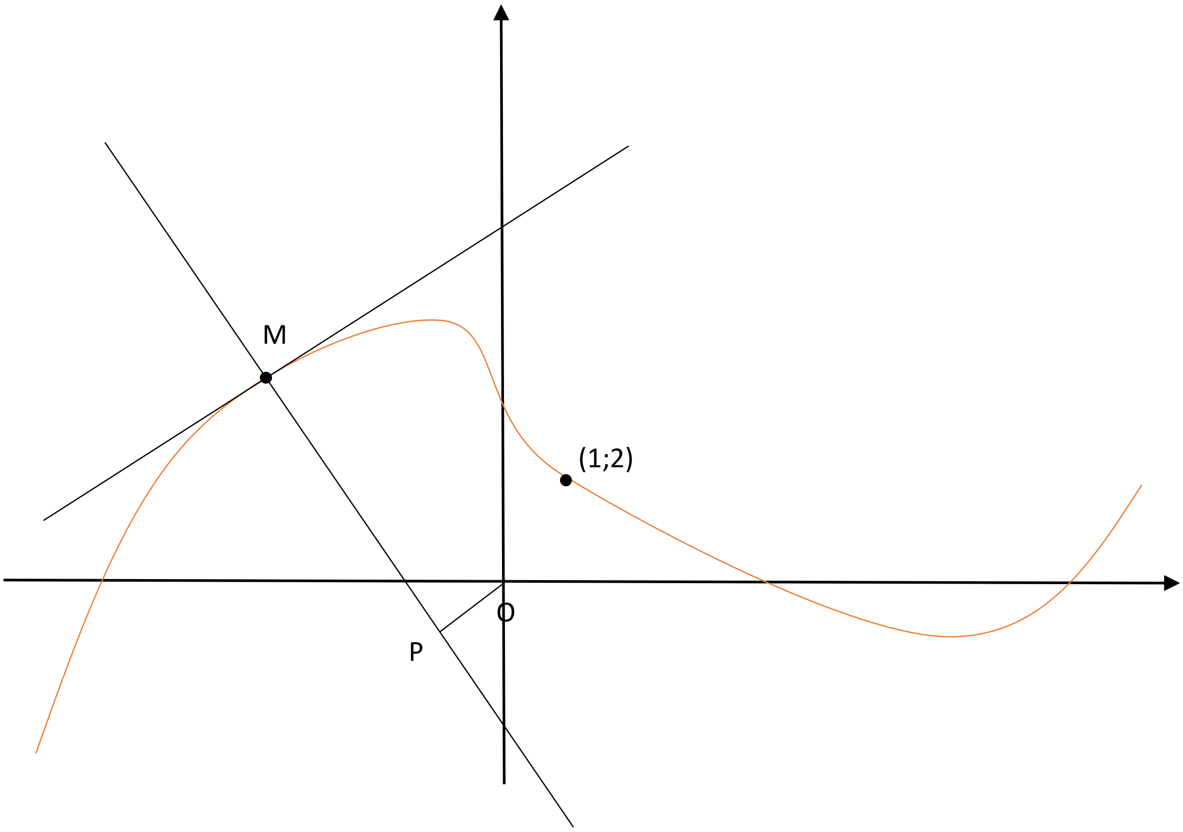
Answer on question 35877 – Math – Differential Calculus

We have a curve passing through (1,2) has the property that the length of the perpendicular drawn from the origin to the normal at any point of the curve is always equal numerically to the ordinate of the point. Please find its equation.

Solution



Let our curve is $y = f(x)$, and we know that $f(1) = 2$.

Let $M(x_0; y_0)$ is an arbitrary point on our curve. Then the distance between the origin and the normal at this point equals y_0. Let us find this distance.

The equation of the normal we will search as $y = kx + b$. The slope of the normal equals $k = -\frac{1}{f'(x_0)}$ (because the slope of the tangent is $f'(x_0)$ and the normal is perpendicular to the tangent. From the condition of two perpendicular lines we get the slope of the normal). Also we know that this line passes through the point M, therefore, we get

$$y_0 = -\frac{1}{f'(x_0)}x_0 + b$$

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$$b = y_0 + \frac{1}{f'(x_0)}x_0$$

And equation of the normal is

$$y = -\frac{1}{f'(x_0)}x + y_0 + \frac{1}{f'(x_0)}x_0.$$

Now let us find the equation of PO. This line passes through the origin and perpendicular to the normal this means that its equation is $y = f'(x_0)x$. We need to find the coordinates of the point P now.

$$-\frac{1}{f'(x_0)}x + y_0 + \frac{1}{f'(x_0)}x_0 = f'(x_0)x$$

And

$$x = \frac{f'(x_0)y_0 + x_0}{\left(f'(x_0)\right)^2 + 1}, \qquad y = \frac{f'(x_0)y_0 + x_0}{\left(f'(x_0)\right)^2 + 1}f'(x_0).$$

Now we can find the distance between the normal and the origin

$$|OP| = \sqrt{\left(\frac{f'(x_0)y_0 + x_0}{\left(f'(x_0)\right)^2 + 1}\right)^2 + \left(\frac{f'(x_0)y_0 + x_0}{\left(f'(x_0)\right)^2 + 1}f'(x_0)\right)^2} = \frac{f'(x_0)y_0 + x_0}{\sqrt{\left(f'(x_0)\right)^2 + 1}}.$$

And this distance equals y_0. As we consider the arbitrary point M then we can denote its coordinates as (x, y). We obtain the equation

$$\frac{f'(x)y + x}{\sqrt{\left(f'(x)\right)^2 + 1}} = y$$

Transforming this equation and remember that $y(x) = f(x)$ we get

$$2xyy' = x^2 + y^2$$

This is the first order differential equation.

$$2x\frac{dy}{dx}y - y^2 = x^2$$

Divide both sides by x we get

$$2\frac{dy}{dx}y - \frac{y^2}{x} = x$$

Let $v(x) = y^2(x)$, which gives $\frac{dv(x)}{dx} = 2y\frac{dy}{dx}$

$$\frac{dv(x)}{dx} - \frac{v(x)}{x} = x$$

Divide both sides by x

$$\frac{1}{x}\frac{dv(x)}{dx} - \frac{v(x)}{x^2} = 1$$

Substitute $-\frac{1}{x^2} = \frac{d}{dx}\left(\frac{1}{x}\right)$

$$\frac{1}{x}\frac{dv(x)}{dx} + v(x)\frac{d}{dx}\left(\frac{1}{x}\right) = 1$$

Apply the inverse product rule to the left-hand side we get

$$\frac{d}{dx}\left(\frac{v(x)}{x}\right) = 1$$

Integrate both sides with respect to x we obtain

$$\frac{v(x)}{x} = x + c$$

Multiply both sides by x we get

$$v(x) = x^2 + cx$$

Inverse substitution

$$y(x) = \sqrt{x^2 + cx}$$

Also we have the initial condition $y(1) = 2$

$$y(1) = \sqrt{1 + c} \quad \Rightarrow \quad c = 3$$

Answer:

$$y(x) = \sqrt{x^2 + 3x}.$$