Integration of squareroot tanx dx?

Solution:

$We\ need\ to\ find \int \sqrt{\tan x}\, dx$

First, we can make a $u-$substitution. Let $u = \sqrt{\tan x}$. Then $du = \frac{\sec^2 x dx}{2\sqrt{\tan x}} = \frac{dx}{2\cos^2 x\sqrt{\tan x}}$.
Recall , $\cos^2 x + \sin^2 x = 1$. If we factor out a $\cos^2 x$ on the left side, we get $(\cos^2 x)(1 + \tan^2 x) = 1$. Now we can rewrite the integral:

$$\int \sqrt{\tan x}\, dx = \int \frac{\sqrt{\tan x}}{(\cos^2 x)(1+\tan^2 x)} dx$$

Now, in integrand we multiply the nominator and the denominator by $2\sqrt{\tan x}$. After doing this, we get:

$$\int \sqrt{\tan x}\, dx = \int \frac{2\tan x}{(2\cos^2 x)\sqrt{\tan x}\,(1+\tan^2 x)} dx$$

$2\tan x = 2u^2$ and $1 + \tan^2 x = 1 + u^4$. If we rewrite the integrand in terms of variable u, we get:

$$\int \sqrt{\tan x}\, dx = \int \frac{2u^2}{1+u^4} du$$

Then, we can factor $1 + u^4$ as:

$$1 + u^4 = \left(u^2 + u\sqrt{2} + 1\right)\left(u^2 - u\sqrt{2} + 1\right);$$

Now we can rewrite the integral:

$$\int \sqrt{\tan x}\, dx = \int \frac{2u^2}{\left(u^2 + u\sqrt{2} + 1\right)\left(u^2 - u\sqrt{2} + 1\right)} du$$

Now, we have to use partial fractions:

$$\frac{2u^2}{\left(u^2 + u\sqrt{2} + 1\right)\left(u^2 - u\sqrt{2} + 1\right)} = \frac{Au + B}{u^2 + u\sqrt{2} + 1} + \frac{Cu + D}{u^2 - u\sqrt{2} + 1}$$

In the right side we reduce terms to a common denominator. Then we go through and equate the coefficients of each power of u in the left side and in the right side so that we can solve for $A,\ B,\ C$, and D.

$$\frac{2u^2}{\left(u^2 + u\sqrt{2} + 1\right)\left(u^2 - u\sqrt{2} + 1\right)} = \frac{Au + B}{u^2 + u\sqrt{2} + 1} + \frac{Cu + D}{u^2 - u\sqrt{2} + 1}$$

$$\Longrightarrow \frac{2u^2}{\left(u^2 + u\sqrt{2} + 1\right)\left(u^2 - u\sqrt{2} + 1\right)} = \frac{(Au+B)\left(u^2 - u\sqrt{2} + 1\right) + (Cu+D)\left(u^2 + u\sqrt{2} + 1\right)}{\left(u^2 + u\sqrt{2} + 1\right)\left(u^2 - u\sqrt{2} + 1\right)}$$

$$\Longrightarrow 2u^2 = (Au+B)\left(u^2 - u\sqrt{2} + 1\right) + (Cu+D)\left(u^2 + u\sqrt{2} + 1\right)$$

$$\Longrightarrow 2u^2 = Au^3 - Au^2\sqrt{2} + Au + Bu^2 - Bu\sqrt{2} + B + Cu^3 + Cu^2\sqrt{2} + Cu + Du^2 + Du\sqrt{2} + D$$

$$\Longrightarrow 2u^2 = (A+C)u^3 + \left(-A\sqrt{2} + B + C\sqrt{2} + D\right)u^2 + \left(A - B\sqrt{2} + C + D\sqrt{2}\right)u + (B+D)$$

Equations derived from coefficients near similar terms:

$$\begin{cases} A + C = 0 (I) \\ -A\sqrt{2} + B + C\sqrt{2} + D = 2 \quad (II) \\ A - B\sqrt{2} + C + D\sqrt{2} = 0 \quad (III) \\ B + D = 0 \quad (IV) \end{cases}$$

Equations (I) and (IV) tell us that $C = -A$ and $D = -B$. Using that information, we replace the expressions in equations (II) and (III). This gives us the following:

$$-A\sqrt{2} + B + (-A)\sqrt{2} + (-B) = 2 \Longrightarrow A = -\frac{\sqrt{2}}{2}$$
$$A - B\sqrt{2} + (-A) + (-B)\sqrt{2} = 0 \Longrightarrow B = 0$$

So, we see that $C = \frac{\sqrt{2}}{2}$ and $D = 0$. Now, we go back to our partial fractions and fill in A, B, C, and D. Doing this, we have the following:

$$\frac{2u^2}{\left(u^2 + u\sqrt{2} + 1\right)\left(u^2 - u\sqrt{2} + 1\right)} = \frac{-\left(\frac{\sqrt{2}}{2}\right)u + 0}{u^2 + u\sqrt{2} + 1} + \frac{\left(\frac{\sqrt{2}}{2}\right)u + 0}{u^2 - u\sqrt{2} + 1} =$$
$$= \frac{-\frac{\sqrt{2}}{2}u}{u^2 + u\sqrt{2} + 1} + \frac{\frac{\sqrt{2}}{2}u}{u^2 - u\sqrt{2} + 1}$$

Finally, returning to the original problem, we can rewrite the integral as follows:

$$\int \sqrt{\tan x}\, dx = \int \left(\frac{-\frac{\sqrt{2}}{2}u}{u^2 + u\sqrt{2} + 1} + \frac{\frac{\sqrt{2}}{2}u}{u^2 - u\sqrt{2} + 1}\right) du$$

Now, we can break $\int \sqrt{\tan x}\, dx$ into two different integrals. Also, we can rewrite the numerator of both integrals. Doing so will allow us to continue our integrations.

Notice that $\frac{\sqrt{2}}{2}u = \frac{\sqrt{2}}{4}\left(2u - \sqrt{2}\right) + \frac{1}{2}$ and likewise that $-\frac{\sqrt{2}}{2}u = -\frac{\sqrt{2}}{4}\left(2u + \sqrt{2}\right) + \frac{1}{2}$. The reason for this will become evident shortly. Here is the first part of the integration, broken into two separate pieces.

$\int \frac{-\frac{\sqrt{2}}{2}u}{u^2+u\sqrt{2}+1} du = \int \frac{-\frac{\sqrt{2}}{4}(2u+\sqrt{2})}{u^2+u\sqrt{2}+1} du + \int \frac{\frac{1}{2}}{u^2+u\sqrt{2}+1} du$

Now, let us consider the first of these two integrals: $\int \frac{-\frac{\sqrt{2}}{4}(2u+\sqrt{2})}{u^2+u\sqrt{2}+1} du$

Here, the substitution is actually straightforward because of the above work. Let $\nu = u^2 +$

$+u\sqrt{2} + 1$. Then that means that $d\nu = (2u + \sqrt{2})du$, which is precisely what we have above. (The constant can be factored out.) That means that we have:

$\int \frac{-\frac{\sqrt{2}}{4}(2u+\sqrt{2})}{u^2+u\sqrt{2}+1} du = -\frac{\sqrt{2}}{4}\int \frac{d\nu}{\nu} = -\frac{\sqrt{2}}{4}\ln|\nu| + C_1 = -\frac{\sqrt{2}}{4}\ln\left|u^2 + u\sqrt{2} + 1\right| + C_1$

Second integral: $\int \frac{\frac{1}{2}}{u^2+u\sqrt{2}+1}du$

$$\frac{\frac{1}{2}}{u^2+u\sqrt{2}+1} = \frac{1}{2\left(u^2+u\sqrt{2}+1\right)} = \frac{1}{2u^2+2u\sqrt{2}+2} = \frac{1}{2u^2+2u\sqrt{2}+1+1} =$$

$$= \frac{1}{\left(u\sqrt{2}+1\right)^2+1} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\left(u\sqrt{2}+1\right)^2+1} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{\left(u\sqrt{2}+1\right)^2+1}$$

That means that we have the following:

$$\int \frac{\frac{1}{2}}{u^2+u\sqrt{2}+1}du = \frac{\sqrt{2}}{2}\int \frac{\sqrt{2}}{\left(u\sqrt{2}+1\right)^2+1}du$$

We can finally integrate further, if we use another substitution. Let $v = u\sqrt{2}+1$, then

$dv = \sqrt{2}du$. Using that substitution, we have:
$\frac{\sqrt{2}}{2}\int \frac{\sqrt{2}}{\left(u\sqrt{2}+1\right)^2+1}du = \frac{\sqrt{2}}{2}\int \frac{dv}{v^2+1} = \frac{\sqrt{2}}{2}tan^{-1}v + C_2.$

Evaluating this, we have

$$\int \frac{\frac{1}{2}}{u^2+u\sqrt{2}+1}du = \frac{\sqrt{2}}{2}tan^{-1}\left(u\sqrt{2}+1\right) + C_2$$

Putting this all together, we have done half of the integration. So far, we have shown:

$$\int \frac{-\frac{\sqrt{2}}{2}u}{u^2+u\sqrt{2}+1} = -\left(-\frac{\sqrt{2}}{4}ln\left|u^2+u\sqrt{2}+1\right| + C_1\right) + \left(\frac{\sqrt{2}}{2}tan^{-1}\left(u\sqrt{2}+1\right) + C_2\right)$$

Now, thankfully, something similar happens for $\int \frac{\frac{\sqrt{2}}{2}u}{u^2-u\sqrt{2}+1}du$.We rewrite $\int \frac{\frac{\sqrt{2}}{2}u}{u^2-u\sqrt{2}+1}du$ as $\int \frac{\frac{\sqrt{2}}{4}(2u-\sqrt{2})}{u^2-u\sqrt{2}+1}du + \int \frac{\frac{1}{2}}{u^2-u\sqrt{2}+1}du.$

For the first integral, we again use a substitution. Let $v = u^2 - u\sqrt{2} + 1$. Then that means that

$dv = (2u - \sqrt{2})du$, which is precisely what we have earlier. (The constant can be factored out.) That means that we have:

$$\int \frac{\frac{\sqrt{2}}{4}\left(2u-\sqrt{2}\right)}{u^2-u\sqrt{2}+1}du = \frac{\sqrt{2}}{4}\int \frac{dv}{v} = \frac{\sqrt{2}}{4}ln|v| + C_3 = \frac{\sqrt{2}}{4}ln\left|u^2-u\sqrt{2}+1\right| + C_3$$

And for the second integral, we again use a trick to change $\int \frac{\frac{1}{2}}{u^2-u\sqrt{2}+1} du$ into something more manageable. Following the steps from above, we see that:

$$\int \frac{\frac{1}{2}}{u^2 - u\sqrt{2} + 1} du = \frac{\sqrt{2}}{2} \int \frac{\sqrt{2}}{\left(u\sqrt{2} - 1\right)^2 + 1} du$$

Again, we use a substitution. Let $v = u\sqrt{2} - 1$. Then that means that $dv = \sqrt{2}du$. That means that we have:

$$\int \frac{\frac{1}{2}}{u^2 - u\sqrt{2} + 1} du = \frac{\sqrt{2}}{2} tan^{-1}\left(u\sqrt{2} - 1\right) + C_4$$

That means we have the following:

$$\int \frac{\frac{\sqrt{2}}{2}u}{u^2 - u\sqrt{2} + 1} du = \left(\frac{\sqrt{2}}{4} ln\left|u^2 - u\sqrt{2} + 1\right| + C_3\right) + \left(\frac{\sqrt{2}}{2} tan^{-1}\left(u\sqrt{2} - 1\right) + C_4\right)$$

And so, putting together everything from above, we see that:

$$\int \sqrt{tan\, x}\, dx = -\left(-\frac{\sqrt{2}}{4} ln\left|u^2 + u\sqrt{2} + 1\right| + C_1\right) + \left(\frac{\sqrt{2}}{2} tan^{-1}\left(u\sqrt{2} + 1\right) + C_2\right) +$$

$$+\left(\frac{\sqrt{2}}{4} ln\left|u^2 - u\sqrt{2} + 1\right| + C_3\right) + \left(\frac{\sqrt{2}}{2} tan^{-1}\left(u\sqrt{2} - 1\right) + C_4\right)$$

Now, we can simplify the above expression just a bit. We can factor $\frac{\sqrt{2}}{2}$ out of all terms. Also, we can use a property of logs to simplify the two natural log expressions. Doing so, we get the following expression:

$$\int \sqrt{tan\, x}\, dx = \frac{\sqrt{2}}{2}\left(tan^{-1}\left(u\sqrt{2} + 1\right) + tan^{-1}\left(u\sqrt{2} - 1\right) + \frac{1}{2} ln\left|\frac{u^2 - u\sqrt{2} + 1}{u^2 + u\sqrt{2} + 1}\right|\right) + C$$

Lastly, we replace u with its expression from the first page. Doing this, we get our final answer:

$\int \sqrt{tan\, x}\, dx = \frac{\sqrt{2}}{2}\left(tan^{-1}\left(\sqrt{2\, tan\, x} + 1\right) + tan^{-1}\left(\sqrt{2\, tan\, x} - 1\right) + \frac{1}{2} ln\left|\frac{tan\, x - \sqrt{2\, tan\, x} + 1}{tan\, x + \sqrt{2\, tan\, x} + 1}\right|\right) + C,$

where C is an arbitrary real constant.

Answer:

$\int \sqrt{tan\, x}\, dx = \frac{\sqrt{2}}{2}\left(tan^{-1}\left(\sqrt{2\, tan\, x} + 1\right) + tan^{-1}\left(\sqrt{2\, tan\, x} - 1\right) + \frac{1}{2} ln\left|\frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1}\right|\right) + \mathrm{C}.$