Divide 15 into two parts such that the square of one no. multiplied with the cube of other no. is maximum

Solution:

Denote x is first number and y is second number. So we have

$$f(x,y) = x^2y^3 \to max$$

and

$$x + y = 15$$

or

$$x = 15 - y.$$

Thus we have

$$f(y) = (15 - y)^2y^3 \to max.$$

We must find the first derivative of $f(y)$

$$f'(y) = -2(15 - y)y^3 + 3(15 - y)^2y^2 = (15 - y)y^2(-5y + 45).$$

So

$$(15 - y)y^2(-5y + 45) = 0,$$

$$15 - y = 0 \ \ or \ \ y^2 = 0 \ \ or \ \ (-5y + 45) = 0,$$

$$y_1 = 15, y_{2,3} = 0, y_4 = 9.$$

We must find the second derivative of $f(y)$

$$f''(y) = -y^2(-5y + 45) + 2y(15 - y)(-5y + 45) - 5(15 - y)y^2$$

and

$$f''(y_1) = f''(15) = -15^2(-75 + 45) + 30(15 - 15)(-75 + 45) - 5(15 - 15)15^2 =$$

$$= -15^2(-30) > 0;$$

$$f''\left(y_{2,3}\right) = f''(0) = -0^2(0 + 45) + 0 \cdot (15 - 0)(0 + 45) - 5(15 - 0) \cdot 0^2 = 0;$$

$$f''(y_4) = f''(9) = -9^2(-45 + 45) + 18(15 - 9)(-45 + 45) - 5(15 - 9) \cdot 9^2 =$$

$$= -30 \cdot 9^2 < 0.$$

Because only $f''(9) < 0$ then $y = 9$ is the only point of maximum. Also we have

$$x = 15 - y = 15 - 9 = 6.$$

Answer: 6 and 9.