Question 35733

We are given $x+\frac{1}{x}=2\cos\frac{\pi}{10}$ and need to find $x^5+\frac{1}{x^5}$.

Let us expand $\left(x+\frac{1}{x}\right)^5=\frac{1}{x^5}+\frac{5}{x^3}+\frac{10}{x}+10x+5x^3+x^5$ and $5\cdot\left(x+\frac{1}{x}\right)^3=\frac{5}{x^3}+5x^3+\frac{15}{x}+15x$. From first expresion , $\frac{1}{x^5}+x^5=\left(x+\frac{1}{x}\right)^5-\left(\frac{5}{x^3}+\frac{10}{x}+10x+5x^3\right)$. The expression in the last brackets on the right might be rewritten as $\frac{5}{x^3}+\frac{10}{x}+10x+5x^3=5\left(x+\frac{1}{x}\right)^3-5\left(\frac{1}{x}+x\right)$. Hence, knowing that $x+\frac{1}{x}=2\cos\frac{\pi}{10}$, obtain:

$$\frac{1}{x^5}+x^5=\left(x+\frac{1}{x}\right)^5-5\left(x+\frac{1}{x}\right)^3+5\left(\frac{1}{x}+x\right)=32\cos^5\frac{\pi}{10}-40\cos^3\frac{\pi}{10}+10\cos\frac{\pi}{10}$$.