Solve the set of linear equations by the matrix method :

$$\begin{cases} a+3b+2c=3 \\ 2a-b-3c=-8 \\ 5a+2b+c=9 \end{cases}$$

We need find vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ such as

A $* \begin{pmatrix} a \\ b \\ c \end{pmatrix} = B$, (1)

where $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & -3 \\ 5 & 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 \\ -8 \\ 9 \end{pmatrix}$.

Multiply (1) by A^{-1}:

$$A^{-1}A * \begin{pmatrix} a \\ b \\ c \end{pmatrix} = A^{-1}B$$

$$E * \begin{pmatrix} a \\ b \\ c \end{pmatrix} = A^{-1}B$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = A^{-1}B$$

1) First find the $\mathbf{det}\,\boldsymbol{A}$

$$\det A = 1*(-1+6) - 2*(3-4) + 5*(-9+2) = 5+2-35 = -28$$

2) **Calculating Matrix of Minors.**

It is easy for example if we need find cell M_{11}, I ignore the values in the current row and columns, and calculate the determinant using the remaining values

$$M_{11} = \begin{pmatrix} * & & \\ & -1 & -3 \\ & 2 & 1 \end{pmatrix} = \begin{vmatrix} -1 & -3 \\ 2 & 1 \end{vmatrix} = -1*1-(-3)*2 = 5$$

$$M=\begin{pmatrix} \begin{vmatrix}-1 & -3\\ 2 & 1\end{vmatrix} & \begin{vmatrix}2 & -3\\ 5 & 1\end{vmatrix} & \begin{vmatrix}2 & -1\\ 5 & 2\end{vmatrix}\\ \begin{vmatrix}3 & 2\\ 2 & 1\end{vmatrix} & \begin{vmatrix}1 & 2\\ 5 & 1\end{vmatrix} & \begin{vmatrix}1 & 3\\ 5 & 2\end{vmatrix}\\ \begin{vmatrix}3 & 2\\ -1 & -3\end{vmatrix} & \begin{vmatrix}1 & 2\\ 2 & -3\end{vmatrix} & \begin{vmatrix}1 & 3\\ 2 & -1\end{vmatrix} \end{pmatrix}$$

$$=\begin{pmatrix} -1*1-(-3)*2 & 2*1-(-3)*5 & 2*2-(-1)*5\\ 3*1-2*2 & 1*1-2*5 & 1*2-5*3\\ 3*(-3)-2*(-1) & -3*1-2*2 & 1*(-1)-2*3 \end{pmatrix}$$

$$=\begin{pmatrix} 5 & 17 & 9\\ -1 & -9 & -13\\ -7 & -7 & -7 \end{pmatrix}$$

3) Matrix of Cofactors

Then find Matrix of Cofactors Just apply a "checkerboard" of minuses to the "Matrix of Minors". In other words, you need to change the sign of alternate cells, like this:

$$\begin{pmatrix} 5 & 17 & 9\\ -1 & -9 & -13\\ -7 & -7 & -7 \end{pmatrix} \sim \begin{pmatrix} + & - & +\\ - & + & -\\ + & - & + \end{pmatrix} \sim \begin{pmatrix} 5 & -17 & 9\\ 1 & -9 & 13\\ -7 & 7 & -7 \end{pmatrix}$$

$$Matrix\ of\ Cofactors=\begin{pmatrix} 5 & -17 & 9\\ 1 & -9 & 13\\ -7 & 7 & -7 \end{pmatrix}$$

4) Adjoint Matrix .

Now "Transpose" all elements of the previous matrix. In other words swap their positions over the diagonal (the diagonal stays the same)

$$Adjoint\ Matrix=\begin{pmatrix} 5 & 1 & -7\\ -17 & -9 & 7\\ 9 & 13 & -7 \end{pmatrix}$$

5) Inverse matrix

And now multiply the Adjoint Matrix by 1/Determinant, so the inverse matrix will be write as

$$A^{-1}=\det A * Adjoint\ Matrix=-\frac{1}{28}\begin{pmatrix} 5 & 1 & -7\\ -17 & -9 & 7\\ 9 & 13 & -7 \end{pmatrix}$$

Solution can be find as:

$$\begin{pmatrix} a\\ b\\ c \end{pmatrix}=A^{-1}*B=-\frac{1}{28}\begin{pmatrix} 5 & 1 & -7\\ -17 & -9 & 7\\ 9 & 13 & -7 \end{pmatrix}*\begin{pmatrix} 3\\ -8\\ 9 \end{pmatrix}=$$

$$= -\frac{1}{28}\begin{pmatrix} 5 * 3 + 1 * (-8) + (-7) * 9 \\ (-17) * 3 + (-9) * (-8) + 7 * 9 \\ 9 * 3 + 13 * (-8) + (-7) * 9 \end{pmatrix} = -\frac{1}{28}\begin{pmatrix} -56 \\ 84 \\ -140 \end{pmatrix} = \begin{pmatrix} -\frac{1}{28} * (-56) \\ -\frac{1}{28} * 84 \\ -\frac{1}{28} * (-140) \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

Answer: $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$