Task:

On a tiny remote island where the death sentence still exists a man can be granted mercy after receiving the death sentence

He is given 18 red balls and 6 green balls

He must divide them between 3 boxes with at least one ball in each

He is blindfolded and must choose a box at random and then a single ball from within the box

He only receives mercy if a red ball is chosen

Find the probability that he receives mercy when he distributes the balls in the most favorable manner.

Solution:

Suppose that a man placed r_1, r_2 and r_3 red balls in the 1^{st}, 2^{nd} and 3^{rd} boxes respectively. And g_1, g_2, g_3 green balls in the 1^{st}, 2^{nd} and 3^{rd} boxes respectively (g_i, r_i are non-negative whole numbers). We have that

(1) $r_1 + r_2 + r_3=18$,

(2) $g_1 + g_2 + g_3=6$.

(3) $r_i+ g_i>0$, i=1, 2, 3.

Denote a valid distribution as d, d=d(r_1, r_2, r_3, g_1, g_2, g_3) is defined by r_1, r_2, r_3, g_1, g_2, g_3, for which (1)-(3) are right. The probability that a man receives mercy is $P(d)=\frac{1}{3}\frac{r_1}{r_1+g_1}+\frac{1}{3}\frac{r_2}{r_2+g_2}+\frac{1}{3}\frac{r_3}{r_3+g_3}=\frac{1}{3}(\frac{r_1}{r_1+g_1}+\frac{r_2}{r_2+g_2}+\frac{r_3}{r_3+g_3})$.

Call the balls distribution d_0 optimal if $P(d_0)=\max_d P(d)=P_0$.

Then

1) $P_0\geq\frac{10}{11}$. **Proof:** $P_0\geq P\left(d(16,1,1,6,0,0)\right)=\frac{1}{3}\left(\frac{16}{16+6}+\frac{1}{1+0}+\frac{1}{1+0}\right)=\frac{1}{3}\left(\frac{8}{11}+2\right)=\frac{1}{3}\frac{30}{11}=\frac{10}{11}$.
2) $r_1, r_2, r_3\geq 1$, because otherwise $P(d_0)\leq\frac{1}{3}(1+1+0)=\frac{2}{3}<\frac{10}{11}$.
3) For $1<a\leq b$, $1<c\leq d$, $\frac{a}{b}+\frac{c}{d}\leq\frac{a+c-1}{b+d-1}+1$.

   Proof: inequality is equivalent to $\frac{a}{b}(b+d-1)+\frac{c}{d}(b+d-1)\leq a+b+c+d-2$

   $$a+a\frac{d}{b}-\frac{a}{b}+c+c\frac{b}{d}-\frac{c}{d}\leq a+b+c+d-2$$

   $$\frac{a}{b}(d-1)+\frac{c}{d}(b-1)\leq b+d-2$$

   But $a\leq b$ and $c\leq d$, therefore, $\frac{a}{b}(d-1)+\frac{c}{d}(b-1)\leq d-1+b-1=b+d-2$.

   Therefore, $\frac{a}{b}(d-1)+\frac{c}{d}(b-1)\leq b+d-2$ is right and $\frac{a}{b}+\frac{c}{d}\leq\frac{a+c-1}{b+d-1}+1$ is right.
4) Using 3): $P(d)=\frac{1}{3}(\frac{r_1}{r_1+g_1}+\frac{r_2}{r_2+g_2}+\frac{r_3}{r_3+g_3})\leq\frac{1}{3}(\frac{r_1}{r_1+g_1}+\frac{r_2+r_3-1}{r_2+g_2+r_3+g_3-1}+1))\leq\frac{1}{3}(\frac{r_1+r_2+r_3-2}{r_1+g_1+r_2+g_2+r_3+g_3-2}+2)=\frac{1}{3}(\frac{16}{16+6}+2)=\frac{10}{11}$.
5) 1) and 4) $\Rightarrow P_0=\frac{10}{11}$.

Answer: $\frac{10}{11}$.