Answer on Question#35079 – Math – Calculus

Take the integral:

$$\int \frac{1}{(x-1)^2 (x^2+4)}\, dx$$

For the integrand $\frac{1}{(x-1)^2 (x^2+4)}$, use partial fractions:

$$= \int \left(\frac{2x-3}{25(x^2+4)} - \frac{2}{25(x-1)} + \frac{1}{5(x-1)^2} \right) dx$$

Integrate the sum term by term and factor out constants:

$$= \frac{1}{25} \int \frac{2x-3}{x^2+4}\, dx + \frac{1}{5} \int \frac{1}{(x-1)^2}\, dx - \frac{2}{25} \int \frac{1}{x-1}\, dx$$

Expanding the integrand $\frac{2x-3}{x^2+4}$ gives $\frac{2x}{x^2+4} - \frac{3}{x^2+4}$:

$$= \frac{1}{25} \int \left(\frac{2x}{x^2+4} - \frac{3}{x^2+4} \right) dx + \frac{1}{5} \int \frac{1}{(x-1)^2}\, dx - \frac{2}{25} \int \frac{1}{x-1}\, dx$$

Integrate the sum term by term and factor out constants:

$$= -\frac{3}{25} \int \frac{1}{x^2+4}\, dx + \frac{2}{25} \int \frac{x}{x^2+4}\, dx + \frac{1}{5} \int \frac{1}{(x-1)^2}\, dx - \frac{2}{25} \int \frac{1}{x-1}\, dx$$

For the integrand $\frac{x}{x^2+4}$, substitute $u = x^2+4$ and $du = 2x\, dx$:

$$= \frac{1}{25} \int \frac{1}{u}\, du - \frac{3}{25} \int \frac{1}{x^2+4}\, dx + \frac{1}{5} \int \frac{1}{(x-1)^2}\, dx - \frac{2}{25} \int \frac{1}{x-1}\, dx$$

Factor 4 from the denominator:

$$= \frac{1}{25} \int \frac{1}{u}\, du - \frac{3}{25} \int \frac{1}{4\left(\frac{x^2}{4}+1\right)}\, dx + \frac{1}{5} \int \frac{1}{(x-1)^2}\, dx - \frac{2}{25} \int \frac{1}{x-1}\, dx$$

Factor out constants:

$$= \frac{1}{25} \int \frac{1}{u}\, du - \frac{3}{100} \int \frac{1}{\frac{x^2}{4}+1}\, dx + \frac{1}{5} \int \frac{1}{(x-1)^2}\, dx - \frac{2}{25} \int \frac{1}{x-1}\, dx$$

For the integrand $\frac{1}{\frac{x^2}{4}+1}$, substitute $s = \frac{x}{2}$ and $ds = \frac{1}{2}\, dx$:

$$= -\frac{3}{50} \int \frac{1}{s^2+1}\, ds + \frac{1}{25} \int \frac{1}{u}\, du + \frac{1}{5} \int \frac{1}{(x-1)^2}\, dx - \frac{2}{25} \int \frac{1}{x-1}\, dx$$

The integral of $\frac{1}{s^2+1}$ is $\tan^{-1}(s)$:

$$= -\frac{3}{50} \tan^{-1}(s) + \frac{1}{25} \int \frac{1}{u}\, du + \frac{1}{5} \int \frac{1}{(x-1)^2}\, dx - \frac{2}{25} \int \frac{1}{x-1}\, dx$$

The integral of $\frac{1}{u}$ is $\log(u)$:

$$= -\frac{3}{50}\tan^{-1}(s) + \frac{\log(u)}{25} + \frac{1}{5}\int \frac{1}{(x-1)^2}\,dx - \frac{2}{25}\int \frac{1}{x-1}\,dx$$

For the integrand $\frac{1}{x-1}$, substitute $p = x - 1$ and $dp = dx$:

$$= -\frac{2}{25}\int \frac{1}{p}\,dp - \frac{3}{50}\tan^{-1}(s) + \frac{\log(u)}{25} + \frac{1}{5}\int \frac{1}{(x-1)^2}\,dx$$

For the integrand $\frac{1}{(x-1)^2}$, substitute $w = x - 1$ and $dw = dx$:

$$= -\frac{2}{25}\int \frac{1}{p}\,dp - \frac{3}{50}\tan^{-1}(s) + \frac{\log(u)}{25} + \frac{1}{5}\int \frac{1}{w^2}\,dw$$

The integral of $\frac{1}{w^2}$ is $-\frac{1}{w}$:

$$= -\frac{2}{25}\int \frac{1}{p}\,dp - \frac{3}{50}\tan^{-1}(s) + \frac{\log(u)}{25} - \frac{1}{5w}$$

The integral of $\frac{1}{p}$ is $\log(p)$:

$$= -\frac{2\log(p)}{25} - \frac{3}{50}\tan^{-1}(s) + \frac{\log(u)}{25} - \frac{1}{5w} + \text{constant}$$

Substitute back for $w = x - 1$:

$$= \frac{-\frac{1}{50}(x-1)\left(4\log(p) + 3\tan^{-1}(s) - 2\log(u)\right) - \frac{1}{5}}{x-1} + \text{constant}$$

Substitute back for $p = x - 1$:

$$= \frac{-\frac{1}{50}(x-1)\left(3\tan^{-1}(s) - 2\log(u) + 4\log(x-1)\right) - \frac{1}{5}}{x-1} + \text{constant}$$

Substitute back for $s = \frac{x}{2}$:

$$= \frac{-\frac{1}{50}(x-1)\left(-2\log(u) + 4\log(x-1) + 3\tan^{-1}\left(\frac{x}{2}\right)\right) - \frac{1}{5}}{x-1} + \text{constant}$$

Substitute back for $u = x^2 + 4$:

$$= \frac{-\frac{1}{50}(x-1)\left(-2\log(x^2+4) + 4\log(x-1) + 3\tan^{-1}\left(\frac{x}{2}\right)\right) - \frac{1}{5}}{x-1} + \text{constant}$$

Which is equal to:

Answer:

$$= \frac{1}{50}\left(2\left(\log(x^2+4) - \frac{5}{x-1} - 2\log(x-1)\right) - 3\tan^{-1}\left(\frac{x}{2}\right)\right) + \text{constant}$$