Answer on Question#35057 – Math – Real Analysis

Question.

Let S and T be non-empty subsets of the real line with s<= t for every s in S and t in T. Show that sup(S)<=inf(T).

Solution.

(a) Suppose that $S \subset R$ and $t = \sup S$. If $r < t$, then there is a number $s \in S$ such that $r < s \leq t$.
(b) Suppose that $S \subset R$ and $t = \inf S$. If $t < r$, then there is a number $s \in S$ such that $t \leq s < r$.

First note that $\sup S$ and $\inf T$ exist, since any $t \in T$ serves as an upper bound for S and any $s \in S$ serves as a lower bound for T. If $\inf T < \sup S$, then by (a) there is an $a \in S$ such that $\inf T < s \leq \sup S$. Since $\inf T < s$, by (b) there is a $t \in T$ such that $\inf \mathrm{T} \leq t < s$, which contradicts the fact that $s \leq t$ for all $s \in S, t \in T$. Hence, it must be the case that $\sup S \leq \inf T$.