Given constants c and d, find the largest and smallest values of cx + dy taken over all points (x,y) of the ellipse x^2/a^2 + y^2/b^2 = 1.

Solution:

First expression: $cx + dy$ (1)

The equation of the ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (2)

First, we can make the change of variables:

Substitute expressions

$$x = au; \quad y = bv \quad (3)$$

in (2) $\Longrightarrow$

$$u^2 + v^2 = 1 \quad (4)$$

Substitute expressions

$$(3) \text{ in } (1):$$

$$cx + dy = ca \cdot u + bd \cdot v$$

Cauchy-Schwarz inequality:

$$ca \cdot u + bd \cdot v \le \sqrt{a^2c^2 + b^2d^2} \cdot \sqrt{u^2 + v^2};$$

In our case

$$\sqrt{u^2 + v^2} = 1$$

then

$$ca \cdot u + bd \cdot v \le \sqrt{a^2c^2 + b^2d^2}$$

Largest value of $cx + dy$:

$$ca \cdot u + bd \cdot v = \sqrt{a^2c^2 + b^2d^2}$$

$$cx + dy = \sqrt{a^2c^2 + b^2d^2}$$

If the point $(x_0; y_0)$ lies on the ellipse (2) then the point $(-x_0; -y_0)$ lies on the ellipse (2) too. If the point $(u_0; v_0)$ lies on the circle (4) then the point $(-u_0; -v_0)$ lies on the circle (4) too.

Smallest value of $cx + dy$ is the largest value with minus sign:

$$cx + dy = -\sqrt{a^2c^2 + b^2d^2}$$

Answer: Largest value: $\sqrt{a^2c^2 + b^2d^2}$

Smallest value: $-\sqrt{a^2c^2 + b^2d^2}$