Find all the functions f (from rational numbers to rational numbers) such that

$f(x+y) + f(x-y) = 2f(x) + 2f(y)$, for all rationales x, y.

Solution.

Find all continuous $f: R \to R$ which satisfy

$$f(x+y) + f(x-y) = 2[f(x) + f(y)] \quad (1).$$

Fix $\delta > 0$ and let $C = \int_0^\delta 2f(y)dy$.

Then

$2\delta f(x) + C = \int_0^\delta 2[f(x) + f(y)]dy = \int_0^\delta (f(x+y) + f(x-y)dy) = \int_x^{x+\delta} f(y)dy + \int_{x-\delta}^{x} f(y)dy = \int_{x-\delta}^{x+\delta} f(y)dy$

.

Now since *f* is continuous, the last expression is a differentiable function of x and thus the first expression must also be differentiable; hence f is differentiable. By induction, f is infinitely differentiable.

Differentiating (1) first with respect to y, we arrive at:

$$f'(x+y) - f'(x-y) = 2f'(y) \quad (2).$$

Differentiating once more with respect to x, we have:

$f''(x+y) = f''(x-y)$,

so f'' is constant.

It follows that $f(x) = ax^2 + bx + c$ are the only potential solutions.

Substituting $x = y = 0$ in (1) and (2) implying $f(0) = f'(0) = 0$.

Hence

$f(x) = ax^2 (a \in R)$

It is easy to check that all such f are indeed solutions.

Answer:

$f(x) = ax^2 (a \in R)$