Answer on Question #34580 – Math – Calculus

Show that the length of the curve $y = \log\sec x$ between the points $x = 0$ and $x = \frac{\pi}{3}$ is $\log(2+\sqrt{3})$.

Solution

First, it is convenient to rewrite the equation of the curve as

$$y = \log\sec x = \ln\frac{1}{\cos x} = \underbrace{\ln 1}_{0} - \ln\cos x = -\ln\cos x ,$$

and rewrite the answer as $\log(2+\sqrt{3}) = \ln(2+\sqrt{3})$.

The length of the curve L between two points a and b is defined as $L = \int_a^b \sqrt{1+(y'(x))^2}\,dx$.

Step 1. Let's calculate the derivative $y'(x)$:

$$y'(x) = (-\ln\cos(x))' = -\frac{1}{\cos x}(-\sin x) = \frac{\sin x}{\cos x}.$$

Step 2. Now let's find $\sqrt{1+(y'(x))^2}$:

$$\sqrt{1+(y'(x))^2} = \sqrt{1+\frac{\sin^2 x}{\cos^2 x}} = \sqrt{\frac{\cos^2 x + \sin^2 x}{\cos^2 x}} = \sqrt{\frac{1}{\cos^2 x}} = \frac{1}{\cos x}.$$

So, the length of the curve is: $L = \int_0^{\frac{\pi}{3}} \sqrt{1+(y'(x))^2}\,dx = \int_0^{\frac{\pi}{3}} \frac{1}{\cos x}dx$.

Step 3. Let's evaluate the indefinite integral $I = \int \frac{1}{\cos x}dx$.

First we can multiply and divide the integrand, i.e. the function that is to be integrated, $\frac{1}{\cos x}$ by $\cos x$:

$$I = \int \frac{1}{\cos x}dx = \int \frac{\cos x}{\cos^2 x}dx = \left|\begin{matrix}\text{Pythagorean identity}\\ \cos^2 x = 1-\sin^2 x\end{matrix}\right| = \int \frac{\cos x}{1-\sin^2 x}dx =$$

$$= \left|\begin{matrix} t = \sin x \\ dt = \cos x dx\end{matrix}\right| = \int \frac{dt}{(1-t^2)}$$

Step 4. Let's use the Partial Fraction Decomposition method.

1) Factor the denominator: $(1-t^2) = (1-t)\cdot(1+t)$
2) Write one partial fraction for each of those factors:
$$\frac{1}{(1-t^2)} = \frac{A}{(1-t)} + \frac{B}{(1+t)} = \frac{A(1+t)+B(1-t)}{(1-t^2)}$$
3) Multiply through by the denominator so we no longer have fractions: $A(1+t)+B(1-t) = 1$
4) Find unknown coefficients A and B. Substituting the roots ("zeros") of the denominator we will

obtain: $\begin{cases} t=1 & A(1+1)+B(1-1)=1 \\ t=-1 & A(1+t)+B(1-t)=1\end{cases} \Rightarrow \begin{cases} 2A = 1 \\ 2B = 1\end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} \\ B = \frac{1}{2}\end{cases}$

Step 5.

$I = \int \frac{dt}{(1-t^2)} = \frac{1}{2}\int \frac{dt}{1-t} + \frac{1}{2}\int \frac{dt}{1+t} = -\frac{1}{2}\ln|1-t| + \frac{1}{2}\ln|1+t| + c = \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + c$, where c is the integration constant

Step 6. Now let's apply inverse substitution of variables

$I=\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+c$.

Step 7. Lets evaluate the length of the curve, i.e. evaluate the definite integral:

$$L=\int_0^{\frac{\pi}{3}}\frac{1}{\cos x}dx=\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|\Bigg|_0^{\frac{\pi}{3}}=\frac{1}{2}\ln\left|\frac{1+\sin\frac{\pi}{3}}{1-\sin\frac{\pi}{3}}\right|-\frac{1}{2}\ln\left|\frac{1+\sin 0}{1-\sin 0}\right|=\frac{1}{2}\ln\left|\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\right|-\frac{1}{2}\ln\left|\frac{1+0}{1-0}\right|=$$

$$=\frac{1}{2}\ln\left|\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\right|-\underbrace{\frac{1}{2}\ln\left|\frac{1+0}{1-0}\right|}_{0}=\frac{1}{2}\ln\left|\frac{2+\sqrt{3}}{2-\sqrt{3}}\right|=\frac{1}{2}\ln\left|\frac{(2+\sqrt{3})\cdot(2+\sqrt{3}}{(2-\sqrt{3})\cdot(2+\sqrt{3})}\right|=\frac{1}{2}\ln\left|\frac{(2+\sqrt{3})^2}{4-3}\right|=$$

$$=\frac{1}{2}\cdot 2\ln\left|\frac{(2+\sqrt{3})}{1}\right|=\ln\left|\frac{(2+\sqrt{3})}{1}\right|=\ln(2+\sqrt{3})-\underbrace{\ln 1}_{0}=\ln(2+\sqrt{3}).$$

Answer: the length of the curve $y=\log\sec x$ between the points $x=0$ and $x=\frac{\pi}{3}$ is $\log(2+\sqrt{3})$.