Answer on question 33901 – Math – Number Theory

Find all integral solutions of x^2 +1≅ 1(mod 5^3).

Solution

We have

$$x^2 \equiv 0(mod\ 5^3),$$

Let $f(x) = x^2$

At first consider the equation

$$f(x) = x^2 \equiv 0(mod\ 5)$$

Obviously that the solution of this equation is

$$x \equiv 0(mod\ 5)$$

This is mean that $x = 5t_1, t_1 \in \mathbb{Z}$.

Now consider the equation

$$\frac{f(0)}{5} + f'(0)t_1 = 0 + 0 \equiv 0(mod\ 5)$$

This equation holds for any integer t_1. And we get 5 solutions:

$$t_1 \equiv 0(mod\ 5),$$

$$t_1 \equiv 1(mod\ 5),$$

$$t_1 \equiv 2(mod\ 5),$$

$$t_1 \equiv 3(mod\ 5),$$

$$t_1 \equiv 4(mod\ 5).$$

And for any integer t_2 we obtain

$$t_1 = 5t_2, \qquad x = 25t_2$$

$$t_1 = 5t_2 + 1, \qquad x = 25t_2 + 5$$

$$t_1 = 5t_2 + 2, \qquad x = 25t_2 + 10$$

$$t_1 = 5t_2 + 3, \qquad x = 25t_2 + 15$$

$$t_1 = 5t_2 + 4, \qquad x = 25t_2 + 20$$

Now consider the comparison $f(x) \equiv 0(mod\ 5^3)$ for all these x.

1) For $x = 25t_2$ consider the equation

$$\frac{f(0)}{25} + f'(0)t_2 \equiv 0(mod\ 5)$$

It holds for any integer t_2.

2) For $x = 25t_2 + 5$ consider the equation

$$\frac{f(5)}{25} + f'(5)t_2 \equiv 0(mod\ 5)$$

$$1 + 10t_2 \equiv 0(mod\ 5)$$

$$10t_2 \equiv -1(mod\ 5)$$

This comparison has no solutions.

3) For $x = 25t_2 + 10$ consider the equation

$$\frac{f(10)}{25} + f'(10)t_2 \equiv 0(mod\ 5)$$
$$4 + 20t_2 \equiv 0(mod\ 5)$$
$$20t_2 \equiv -4(mod\ 5)$$

This comparison has no solutions too.

4) For $x = 25t_2 + 15$ consider the equation

$$\frac{f(15)}{25} + f'(15)t_2 \equiv 0(mod\ 5)$$
$$9 + 30t_2 \equiv 0(mod\ 5)$$
$$30t_2 \equiv 9 \equiv 4(mod\ 5)$$

This comparison has no solutions.

5) For $x = 25t_2 + 20$ consider the equation

$$\frac{f(20)}{25} + f'(20)t_2 \equiv 0(mod\ 5)$$
$$16 + 40t_2 \equiv 0(mod\ 5)$$
$$40t_2 \equiv -16 \equiv -1(mod\ 5)$$

This comparison has no solutions.

Therefor we get 5 solutions for t_2:

$$t_2 \equiv 0(mod\ 5),$$
$$t_2 \equiv 1(mod\ 5),$$
$$t_2 \equiv 2(mod\ 5),$$
$$t_2 \equiv 3(mod\ 5),$$
$$t_2 \equiv 4(mod\ 5).$$

Only for the 1) case. And we get

$$t_2 = 5t_3, \qquad x = 125t_3$$
$$t_2 = 5t_3 + 1, \qquad x = 125t_3 + 25$$
$$t_2 = 5t_3 + 2, \qquad x = 125t_3 + 50$$
$$t_2 = 5t_3 + 3, \qquad x = 125t_3 + 75$$
$$t_2 = 5t_3 + 4, \qquad x = 125t_3 + 100$$

Answer: $x \equiv 0(mod\ 125), x \equiv 25(mod\ 125), x \equiv 50(mod\ 125), x \equiv 75(mod\ 125), x \equiv 100(mod\ 125).$