Let x be the number of eggs. If we remove a at a time and find b left over then dividing x by a gives a remainder of b. So $x \equiv b \pmod{a}$. We get:

$$\begin{aligned} x &\equiv 1 \pmod{2} \qquad (1) \\ x &\equiv 2 \pmod{3} \qquad (2) \\ x &\equiv 3 \pmod{4} \qquad (3) \\ x &\equiv 4 \pmod{5} \qquad (4) \\ x &\equiv 5 \pmod{6} \qquad (5) \\ x &\equiv 1 \pmod{7} \qquad (6) \end{aligned}$$

Now to use the Chinese remainder theorem we need the numbers we are modding by to be pairwise relatively prime. So we need to get rid of some of the equations. Whenever (3) is true (1) will automatically be true, so we can drop (1). (5) implies (2), so drop (2). As for equations (3) and (5) we have $\mathrm{lcm}(4,6) = 12$. Then note that from 1 to 12 only the numbers 3, 7, and 11 satisfy (3); 5 and 11 satisfy (5). 11 is the only one that does both so you can replace those two with $(*)$ $x \equiv 11 \pmod{12}$. That gives:

$$\begin{aligned} x &\equiv 4 \pmod{5} \qquad (4) \\ x &\equiv 1 \pmod{7} \qquad (6) \\ x &\equiv 11 \pmod{12} \qquad (*) \end{aligned}$$

Which can be expressed as

$$x \equiv 119 \pmod{420}$$

Answer: the smallest number of eggs is 119.