Task. Let $\varepsilon \geq 0$ and $\delta \geq 0$ and $a \in \mathbb{R}$. Show that $V_\varepsilon(a) \cap V_\delta(a)$ and $V_\varepsilon(a) \cup V_\delta(a)$ are γ neighborhoods of a for appropriate values of γ.

Solution. Recall that by definition of topology in $\mathbb{R}$ a neighbourhood $V_t(a)$ is the interval $(a-t, a+t)$ for each $t > 0$, so it consists of all $x \in \mathbb{R}$ such that

$$a - t < x < a + t$$

which is the same as

$$-t < x - a < t.$$

In particular,

$$V_\varepsilon(a) = (a-\varepsilon, a+\varepsilon) \qquad V_\delta(a) = (a-\delta, a+\delta).$$

1) First we will show that

$$V_\varepsilon(a) \cap V_\delta(a) = V_{\min\{\varepsilon,\delta\}}(a).$$

Indeed, the intersection

$$V_\varepsilon(a) \cap V_\delta(a)$$

consists of all $x \in \mathbb{R}$ for which both of the following pairs of inequalities hold:

$$-\varepsilon < x - a < \varepsilon \qquad \text{and} \qquad -\delta < x - a < \delta.$$

Notice that

$$-\varepsilon < x - a \qquad \text{and} \qquad -\delta < x - a$$

if anf only if

$$-\min\{\varepsilon, \delta\} < x - a.$$

Similarly,

$$x - a < \varepsilon \qquad \text{and} \qquad x - a < \delta$$

if anf only if

$$x - a < \min\{\varepsilon, \delta\}.$$

In other words,

$$V_\varepsilon(a) \cap V_\delta(a)$$

consists of all $x \in \mathbb{R}$ for which

$$-\min\{\varepsilon, \delta\} < x - a < \min\{\varepsilon, \delta\},$$

and so

$$V_\varepsilon(a) \cap V_\delta(a) = V_{\min\{\varepsilon,\delta\}}(a).$$

2) Now we wil prove that

$$V_\varepsilon(a) \cup V_\delta(a) = V_{\max\{\varepsilon,\delta\}}(a).$$

Indeed, the union

$$V_\varepsilon(a) \cup V_\delta(a)$$

consists of all $x \in \mathbb{R}$ for which at least one of following pairs of inequalities hold:

$$-\varepsilon < x - a < \varepsilon \qquad \text{and} \qquad -\delta < x - a < \delta.$$

Notice that at least one the following inequalities hold

$$-\varepsilon < x - a \qquad \text{or} \qquad -\delta < x - a$$

if anf only if

$$-\max\{\varepsilon, \delta\} < x - a.$$

Similarly, at least one the following inequalities hold

$$x - a < \varepsilon \qquad \text{or} \qquad x - a < \delta$$

if anf only if

$$x - a < \max\{\varepsilon, \delta\}.$$

In other words,

$$V_\varepsilon(a) \cap V_\delta(a)$$

consists of all $x \in \mathbb{R}$ for which

$$-\max\{\varepsilon, \delta\} < x - a < \max\{\varepsilon, \delta\},$$

and so

$$V_\varepsilon(a) \cap V_\delta(a) = V_{\max\{\varepsilon,\delta\}}(a).$$

Answer.

$$V_\varepsilon(a) \cap V_\delta(a) = V_{\min\{\varepsilon,\delta\}}(a), \qquad V_\varepsilon(a) \cap V_\delta(a) = V_{\max\{\varepsilon,\delta\}}(a).$$