$$\int_0^{\frac{2}{\pi}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

Solution.

This integral is not evaluated, so compute it up to x. So series expansion for $\sqrt{\cos x}$ at $x=0$:

$$\sqrt{\cos x} = \sqrt{1-\frac{x^2}{2}+\frac{x^4}{24}+O(x^5)} = 1+O(x)$$

Now for $\sqrt{\sin x}$:

$$\sqrt{\sin x} = \sqrt{x-\frac{x^3}{6}+O(x^4)} = \sqrt{x}+O\left(x^{\frac{5}{2}}\right)$$

So calculate this integral:

$$\int_0^{\frac{2}{\pi}} \left(\frac{1}{\sqrt{x}+1}\right)dx$$

Indefinite integral:

$$\int \left(\frac{1}{\sqrt{x}+1}\right)dx$$

For the integrand $\frac{1}{\sqrt{x}+1}$, substitute $u=\sqrt{x}$ and $du=\frac{1}{2\sqrt{x}}dx$:

$$\int \left(\frac{1}{\sqrt{x}+1}\right)dx = 2\int \frac{u}{1+u}du$$

For the integrand $\frac{u}{1+u}$, do long division:

$$= 2\int \left(1-\frac{1}{1+u}\right)du$$

Integrate the sum term by term and factor out constants:

$$= 2\int 1du - 2\int \frac{1}{1+u}du = 2u + 2\ln(1+u)+C$$

Substitute back for $u=\sqrt{x}$

$$= 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right)+C$$

So integrate definite integral:

$$\int_{0}^{\frac{2}{\pi}} \left(\frac{1}{\sqrt{x}+1}\right) dx = 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) \Big|_{0}^{2/\pi} = 2\left(\sqrt{\frac{2}{\pi}} - \ln\left(1+\sqrt{\frac{2}{\pi}}\right)\right) \approx 0.423$$

Answer:

$$\int_{0}^{\frac{2}{\pi}} \left(\frac{1}{\sqrt{x}+1}\right) dx \approx 0.423$$