Find a surface satisfying $r+s=0$ and touching the elliptic paraboloid $z=4x^2+y^2$ along its section by the plane $y=2x+1$.

Solution.

Here the differential equation is $r+s=0$ which can be written as

$$\frac{\partial p}{\partial x}+\frac{\partial q}{\partial x}=0 \text{ or } \frac{\partial}{\partial x}(p+q)=0$$

Integrate with respect to x:

$$p+q=f(y)$$

or

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=f(y)$$

Also from $z=4x^2+y^2$

$$\frac{\partial z}{\partial x}=p=8x$$

$$\frac{\partial z}{\partial y}=q=2y$$

If $r+s=0$ touches $z=4x^2+y^2$ along its section by the plane $y=2x+1$ then the values of p and q for an point on this plane should be equal:

$$p+q=8x+2y=f(y)$$

We have

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=8x+2y$$

Integrate with respect to y:

$$z+y\frac{\partial z}{\partial x}=8xy+y^2$$

From $y=2x+1$:

$$2x+1-y=0$$

Multiply by 4:

$$8x-4y+4=0$$

We have

$$z+4x^2+y^2+8xy+8x-4y+4=0$$

Answer:

$$z+4x^2+y^2+8xy+8x-4y+4=0$$