Let f(x) = e^x cos(x) . The power series expansion of f(x) centered at 0 is f(x) = 1 + x - x^3 /3 - x^4 /6 + c5 x^5 + x^7 / 630 + ...
a) show that c5 = - 1/30
b) estimate integral from 0 to 1 e^x cos(x) dx using the given series.
c) estimate f'(1) using the given series

Solution:

We expand our function in a Taylor series.

The Taylor series of a real or complex-valued function ƒ(x) that is infinitely differentiable at a real or complex number a is the power series

$$f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\cdots.$$

which can be written in the more compact sigma notation as

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

where n! denotes the factorial of n and $f^{(n)}(a)$ denotes the nth derivative of f evaluated at the point a. The derivative of order zero f is defined to be f itself and $(x - a)^0$ and 0! are both defined to be 1. In the case that a = 0, the series is also called a Maclaurin series.

So

$f(x)=e^x\cos(x)\to f(0)=1$

$f'(x)=e^x(\cos(x)-\sin(x))\to f'(0)=1$

$f''(x)=-2e^x\sin(x)\to f''(0)=0$

$f'''(x)=-2e^x(\sin(x)+\cos(x))\to f'''(0)=-2$

$f^{(4)}(x)=-4e^x\cos(\mathrm{x})\to f^{(4)}(0)=-4$

$f^{(5)}(x)=-4e^x(\cos(x)-\sin(x))\to f^{(5)}(0)=-4$

$f(0)=1+x-\frac{1}{3}x^3-\frac{1}{6}x^4-\frac{1}{30}x^5+\frac{1}{630}x^7+..$

b) estimate integral from 0 to 1 e^x cos(x) dx using the given series.

$$\int_0^1 e^x\cos(x)\,dx\approx\int_0^1\left(1+x-\frac{1}{3}x^3-\frac{1}{6}x^4-\frac{1}{30}x^5+\frac{1}{630}x^7\right)dx=1+\frac{1}{2}-\frac{1}{12}-\frac{1}{30}+\frac{1}{5040}$$
$$=1.383532$$

estimate f'(1) using the given series

$f(0)=1+x-\frac{1}{3}x^3-\frac{1}{6}x^4-\frac{1}{30}x^5+\frac{1}{630}x^7+..$

$f'(x)=1-x^2-\frac{2}{3}x^3-\frac{1}{6}x^4+\frac{1}{90}x^6$

$f'(1)=-0.82222$