33275:

Task.

A calculator gives us integral from 0 to 1/2 1/(1+x^7) dx ≈ 0.4995

Use power series to find an answer which is more accurate, correct to at least 7 decimals. Show and explain your work and give an estimate of the error

Solution. We use formula

$\frac{1}{1+y} = 1 - y + y^2 - y^3 + y^4 - \cdots = \sum_{k=0}^{\infty}(-1)^k y^k,\ |y| < 1$

for

$\frac{1}{1+x^7} = 1 - x^7 + x^{14} - x^{21} + x^{28} - \cdots = \sum_{k=0}^{\infty}(-1)^k x^{7k},\ |x| < 1$ (1)

We can integrate (1) within convergence interval $(-1; 1)$.

To estimate initial integral

$\int_0^{1/2} \frac{dx}{1+x^7}$ (2)

we integrate (1), obtain

$\sum_{k=0}^{\infty}(-1)^k \frac{x^{7k+1}}{(7k+1)}$ (3)

The series (3) belongs to the series of Leibniz type that's why the absolute value of the remainder of the series (3) does not exceed the first omitted term. Solve inequality

$\frac{1}{(7k+1)2^{(7k+1)}} < 10^{-7}$

and find that $k \geq 3$, k is integer.

In (3) select summands up to one which is not less than 0.0000001 (as shown above, these are terms where $k = 0,1,2$, all the next summands will decrease), unlike the value from calculator, more exact approximation of (2) is

$$\int_0^{1/2}(1 - x^7 + x^{14} - x^{21} + x^{28})dx = \left(x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \frac{x^{29}}{29}\right)_0^{1/2}$$
$$= \frac{1}{2} - \frac{1}{2^8 8} + \frac{1}{2^{15}15} - \frac{1}{2^{22}22} + \frac{1}{2^{29}29}$$
$$\approx 0.5 - 0.00048828125 + 0.0000020345 - 0.00000001$$
$$\approx 0.4995137$$

The summand $\frac{1}{2^{29}29}$ and all the next ones are less than 10^{-7} that is why we do not calculate them.