Integrate with respect to x: $\int e^{5x}\sin 3x\, dx$

a. $\frac{3}{34}e^{5x}\left(\frac{1}{3}\sin 3x - \cos 3x\right) + c$

b. $\frac{3}{24}e^{5x}\left(\frac{5}{3}\sin 3x - \cos 3x\right) + c$

c. $\frac{3}{34}e^{5x}\left(\frac{5}{3}\sin 3x - \cos 3x\right) + c$ – **yes*** (please let me know if this is correct)

d. $\frac{3}{34}e^{5x}\left(\frac{1}{3}\sin 3x - \cos 2x\right) + c$

Solution.

We must integrate it twice **by parts** using this formula (or theorem):

$$\int u\, dv = vu - \int v\, du$$

Let's find this integral:

$$I = \int e^{5x}\sin 3x\, dx = \left| \begin{matrix} u = \sin 3x & du = d\sin 3x = 3\cos 3x\, dx \\ dv = e^{5x}dx & v = \int e^{5x}dx = \frac{1}{5}e^{5x} \end{matrix} \right| = \frac{1}{5}e^{5x}\sin 3x - \int \frac{1}{5}e^{5x}\cdot 3\cos 3x\, dx =$$

$$= \frac{1}{5}e^{5x}\sin 3x - \frac{3}{5}\int \boldsymbol{e^{5x}\cos 3x\, dx} = \frac{1}{5}e^{5x}\sin 3x - \frac{3}{5}\boldsymbol{I_1}$$

Let's find I_1:

$$\int e^{5x}\cos 3x\, dx = \left| \begin{matrix} u = \cos 3x & du = d\cos 3x = -3\sin 3x\, dx \\ dv = e^{5x}dx & v = \int e^{5x}dx = \frac{1}{5}e^{5x} \end{matrix} \right| = \frac{1}{5}e^{5x}\cos 3x - \int \frac{1}{5}e^{5x}\cdot(-3\sin 3x)dx =$$

$$= \frac{1}{5}e^{5x}\cos 3x + \frac{3}{5}\int e^{5x}\sin 3x\, dx$$

So

$$\boldsymbol{I} = \frac{1}{5}e^{5x}\sin 3x - \frac{3}{5}\boldsymbol{I_1} = \frac{1}{5}e^{5x}\sin 3x - \frac{3}{5}\left(\frac{1}{5}e^{5x}\cos 3x + \frac{3}{5}\int e^{5x}\sin 3x\, dx\right) = \frac{1}{5}e^{5x}\sin 3x - \frac{3}{25}e^{5x}\cos 3x -$$

$$-\frac{9}{25}\int e^{5x}\sin 3x\, dx = \frac{1}{5}e^{5x}\sin 3x - \frac{3}{25}e^{5x}\cos 3x - \frac{9}{25}\boldsymbol{I} = \boldsymbol{I}$$

And now we can find $\boldsymbol{I}$ from the equation:

$$\frac{1}{5}e^{5x}\sin 3x - \frac{3}{25}e^{5x}\cos 3x - \frac{9}{25}I = I$$

$$I\left(1 + \frac{9}{25}\right) = e^{5x}\left(\frac{1}{5}\sin 3x - \frac{3}{25}\cos 3x\right)$$

$$\frac{34}{25}I = e^{5x}\cdot\frac{5\sin 3x - 3\cos 3x}{25} \quad \Big| \times 25$$

$$I = \frac{e^{5x}}{34}(5\sin 3x - 3\cos 3x) + c$$

$$I = \frac{3}{34}e^{5x}\left(\frac{5}{3}\sin 3x - \cos 3x\right) + c$$

Answer: variant c is correct:

c. $\frac{3}{34}e^{5x}\left(\frac{5}{3}\sin 3x - \cos 3x\right) + c$